Teacher Notes

Activity 7

Graphs of Functions and Their Derivatives

Objectives

- See the derivative as an indicator of increasing/decreasing function behavior
- See the derivative as an indicator of local maxima/minima function behavior
- Graphically associate a function with its derivative

Materials

- TI-84 Plus / TI-83 Plus

Teaching Time

- 75 minutes

Abstract

In this activity, the concepts of increasing and decreasing function behavior are defined. This is followed by a graphical and symbolic exploration designed to show the student how and why the derivative can be used as an indicator for this behavior. The concepts of local maxima and minima are then informally defined, followed by questions that allow students to uncover the ideas behind the first derivative test. A series of questions invites students to synthesize these ideas by comparing graphs of functions and their derivatives. Finally, the derivative of a function f is compared to the derivative of $f + h$. This activity deals only with functions that are differentiable for all real numbers (except for one function in the matching exercise).

Management Tips and Hints

Prerequisites

Students should:

- be able to graph and generate tables of functions on the graphing handheld.
- know the definition of the derivative and have experience using derivatives.
- know the derivative of the sine function.

Note: *This activity would make a good precursor to the first derivative test.*

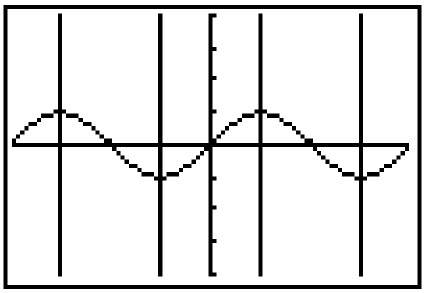
Evidence of Learning

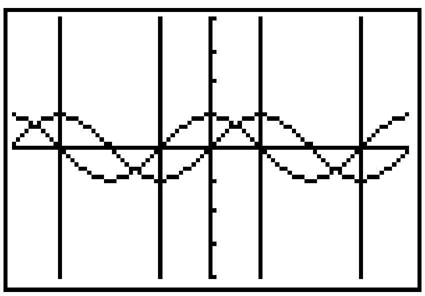
Students should be able to state how the derivative is an indicator of the original function's increasing or decreasing behavior and how the derivative can help find local maxima and minima where differentiable.

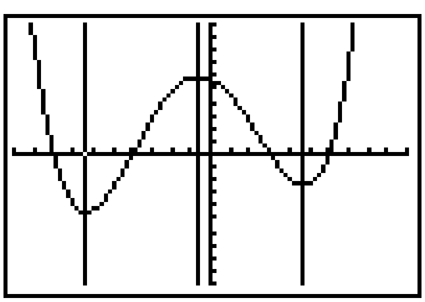
Extensions

After doing this activity, students should be challenged to draw a reasonably accurate graph of the derivative of a function given only the graphical representation of the function. One way to do this is to project a function directly onto a chalkboard using a ViewScreen™ and have a student draw the derivative graph directly on the board. The graphing handheld can then generate the graph of the derivative as a check.

Activity Solutions

1. 

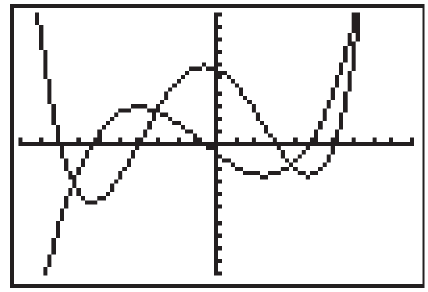
2. 

3. When a function is increasing, the derivative is positive. When a function is decreasing, the derivative is negative.
4. Positive; $f(x + h) > f(x)$ because $(x + h) > x$ (h is positive) and f is increasing.
5. Positive; $f(x + h) - f(x) > 0$ and $h > 0$.
6. Negative; $f(x + h) < f(x)$ because $(x + h) < x$ (h is negative) and f is increasing.
7. Positive; $f(x + h) - f(x) < 0$ and $h < 0$.
8. Positive; $f(x + h) - f(x) > 0$ and $h > 0$.
9. Positive; $f(x + h) - f(x) < 0$ and $h < 0$.
10. 

11. The derivative changes from positive to negative.

12. The derivative changes from negative to positive.

13. Yes



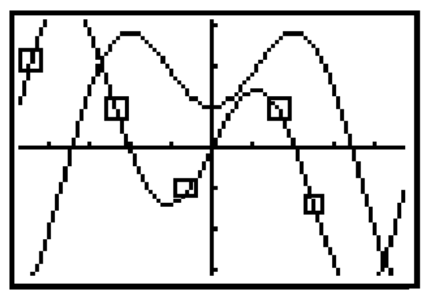
14. 0

15. It changes from negative to positive.

16. It changes from positive to negative.

17. The original function has a local minimum between $^{-}7$ and $^{-}6$ (looking at the graph) and a local maximum between $^{-}1$ and 0. So the results from Questions **15** and **16** are consistent with the answers to Questions **11** and **12**.

18. The graph of the derivative is identified with rectangles.



19. From top to bottom: *B*, *A*, *D*, *C*

20. They have the same shape.

21. The second function is shifted upward by two units.

22. The derivatives of these functions are identical because for any *x*, each function has the same tangent line slope.

$$\frac{d}{dx}f(x) = \frac{d}{dx}(f(x) + k)$$

Students may also know that the derivative of a sum is the sum of the derivatives and that the derivative of a constant function is zero. This type of symbolic reasoning should be connected to the graphical argument and vice versa.

EXPLORATIONS™

Activity 7

Objectives

- See the derivative as an indicator of increasing/decreasing function behavior
- See the derivative as an indicator of local maxima/minima function behavior
- Graphically associate a function with its derivative

Materials

- TI-84 Plus / TI-83 Plus

Graphs of Functions and Their Derivatives

Introduction

When studying relationships between quantities, engineers, scientists, and economists often use information about a rate of change of one quantity relative to a second quantity. In this activity, you will see how a function's rate of change information (in the form of the graph of the derivative) describes many things about the behavior of the function itself.

Exploration

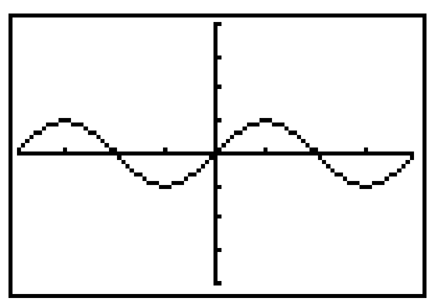
The derivative provides an easy indicator of whether a function is increasing or decreasing. Increasing means the function values are going up as *x* goes up. Decreasing means the function values are going down as *x* goes up.

Suppose that a function *f* is defined on an interval *I*. Let x_1 and x_2 be any real numbers in *I* where $x_1 < x_2$.

The function *f* is increasing over the interval *I* if $f(x_1) \le f(x_2)$.

The function *f* is decreasing over the interval *I* if $f(x_1) \ge f(x_2)$.

The graph of $y = \sin(x)$ is shown.



1. Draw vertical lines on the graph where the function changes from increasing to decreasing and from decreasing to increasing.
2. On your graphing handheld, generate the graphs of $y = \sin(x)$ and $y = \cos(x)$ by selecting **7:Ztrig** in the **ZOOM Menu**. Sketch the graph of $y = \cos(x)$ on the graph of $y = \sin(x)$ on the previous page.
3. Remember that $\cos(x)$ is the derivative of $\sin(x)$. What property of the derivative seems to predict whether the function is increasing or decreasing? Write down the rules that seem to connect the derivative with increasing and decreasing function behavior.

Now you will explore why the sign of f' seems to predict where the function f is increasing or decreasing.

Suppose that f is increasing, h is positive, and f is differentiable for all real numbers.

4. Is $f(x+h)-f(x)$ positive or negative? Why?
5. Is $\frac{f(x+h)-f(x)}{h}$ positive or negative? Why?

Suppose that f is increasing, h is negative, and f is differentiable for all real numbers.

6. Is $f(x+h)-f(x)$ positive or negative? Why?
7. Is $\frac{f(x+h)-f(x)}{h}$ positive or negative? Why?
8. If f is increasing, is $\lim\limits_{h \to 0^+} \frac{f(x+h)-f(x)}{h}$ positive or negative?

   Note: $h \to 0^+$ *means that* h *approaches zero through positive numbers.*

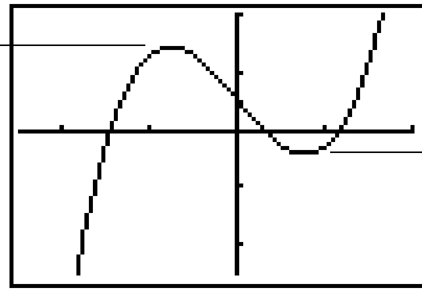
   Explain your reasoning.
9. If f is increasing, is $\lim\limits_{h \to 0^-} \frac{f(x+h)-f(x)}{h}$ positive or negative?

   Note: $h \to 0^-$ *means that* h *approaches zero through negative numbers.*

   Explain your reasoning.

   Similar reasoning can be used to investigate the behavior of f' when f is decreasing.

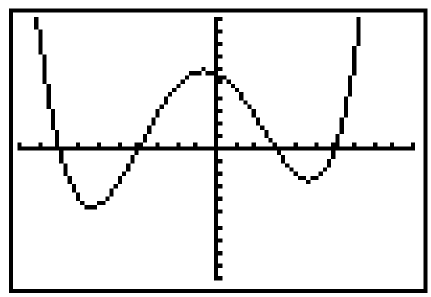
The derivative is also an indicator of whether a function has achieved a local maximum or a local minimum. Graphically, a local maximum is the function value *at the top of a hill* and a local minimum is the function value *at the bottom of a valley*, as the following figure illustrates.



Keep in mind that a function may have many local maxima and local minima.

Input the following expression into **Y1** in the **Y=** editor, and graph it in the standard viewing window: **0.01(x-6)(x-3)(x+4)(x+8)**

Your screen should match the screen shown:



10. Draw vertical lines on the graph where the function changes from increasing to decreasing and from decreasing to increasing.

11. Suppose that a function f has a local maximum at x_M. How does the derivative change as input values change from a little less than x_M to a little more than x_M?

12. Suppose that a function f has a local minimum at x_m. How does the derivative change as input values change from a little less than x_m to a little more than x_m?

13. To see an approximation of the derivative of this function, input the expression **nDeriv(Y1, X, X)** into **Y2** in the **Y=** editor, and graph it along with **Y1** (get nDeriv by selecting **8:nDeriv** in the **MATH Menu**). Draw this derivative approximation on the graph of **Y1** shown above.

Does the graph of the function and its derivative seem consistent with your answers to Questions **11** and **12**?

14. Suppose that a function f is differentiable for all real numbers. What is the value of the derivative at a local maximum or local minimum?

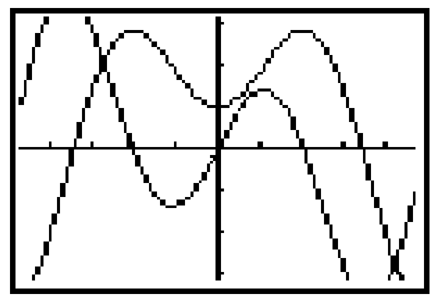
Generate a table with **Y1** and **Y2** selected using the following setup.

15. How does the derivative approximation behave from -7 to -6?

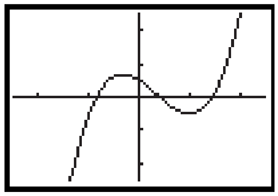
16. How does the derivative approximation behave from -1 to 0?

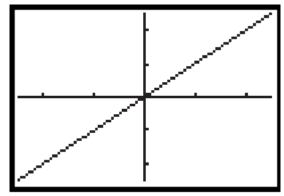
17. Explain how this behavior relates to the discussion so far.

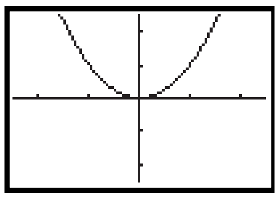
18. The figure shows the graph of a function and its derivative. Which graph is the function, and which is the derivative? Trace over the graph of the derivative with your pencil.

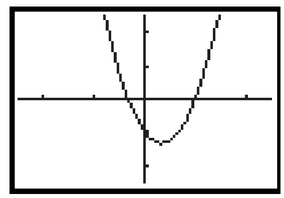


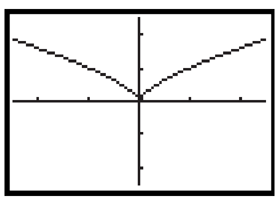
19. Match the graphs of the functions in the first column below with the graphs of their derivatives in the second column below.

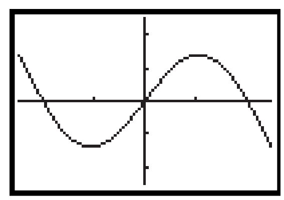
_____ 

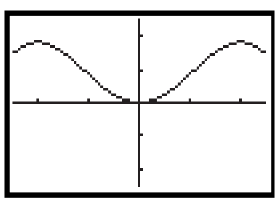
A 

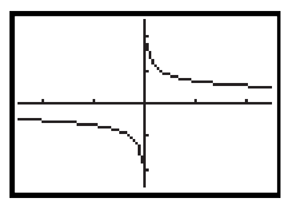
_____ 

B 

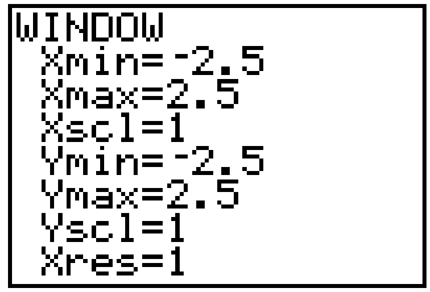
_____ 

C 

_____ 

D 

For the next three questions, work with a partner so that you can compare results on two different graphing handhelds. One person should input the expression $x^3 - 2x^2 - x$ into **Y1**, and the other person should input the expression $x^3 - 2x^2 - x + 2$ into **Y1**. Graph your functions using the viewing window given.



20. How are these functions alike?

21. How are these functions different?

Now graph an approximation of the derivative of your functions by entering the expression **nDeriv(Y1, X, X)** into **Y2** and graphing it together with your function in **Y1**.

22. Write a paragraph that compares the derivatives of these two functions and explains why you got the results that you did. Conclude with a rule that connects the derivative of a function $f(x)$ with the derivative of $f(x) + k$ (where k is a real number).