



nSpire-ing Extraneous Solution Understanding

(Teacher Notes)

by: Jeremy Zelkowski, The University of Alabama

Activity Overview: nSpire-ing conceptual understanding of extraneous solutions of radical equations

Math: Algebra II: Equations

Math: Algebra II: Radical Functions

Math: Algebra II: Transformations

Description	Instructor Notes	Slides/Handouts/Files
<p>This lesson focuses on the conceptual understanding of extraneous solutions to radical function equations. Function transformations, graphical-algebraic function representations, and equation solving are each included.</p>	<p>This lesson is driven by two high-level demanding cognitive task (Stein, Smith, Henningsen, & Silver, 2000) questions pertaining to radical equations. Typically, these two questions are "unanswerable" for students in a classroom where pencil and paper skills are the primary focus concerning solving radical equations. These two problems will drive this lesson and students will have the opportunity to provide solutions to these problems at the end of the lesson. Moreover, this lesson focuses on the construct of flexibility and reversibility in problem solving (Confrey & Lanier, 1980; Rachlin, 1985).</p>	<p>Teacher Notes Word .doc Student Handout Word .doc TI-nSpire CAS .tns file</p>

Teacher Preparation

- Estimated time: 90 minutes to 2 days + homework
- This should not be the first lesson on solving radical equations. Students should have solved basic radical equations algebraically.
- Lesson designed for nSpire CAS classroom in small groups.
- Teachers should work through instructor notes and CAS lesson before implementing in classroom.
- Download teacher notes and student handout .doc files. Download nSpire CAS .tns file for the nSpire CAS.
- Classrooms without full set of nSpire CAS handhelds—Teachers could adapt to classroom lesson with nSpire CAS software.
- Teachers should be familiar with the nSpire CAS

Notes



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Here are two high-level demanding cognitive task (Stein, Smith, Henningsen, & Silver, 2000) questions pertaining to radical equations. Typically, these two questions are "unanswerable" for students in a classroom where pencil and paper skills are the primary focus concerning solving radical equations. These two problems will drive this lesson and students will have the opportunity to provide solutions to these problems at the end of the lesson. Moreover, this lesson focuses on the construct of flexibility and reversibility in problem solving (Confrey & Lanier, 1980; Rachlin, 1985). This lesson aligns with the **NCTM standards for grades 9-12—**

9-12.1.1.2 compare and contrast the properties of numbers and number systems, including the rational and real numbers, and understand complex numbers as solutions to quadratic equations that do not have real solutions

9-12.1.2.1 judge the effects of such operations as multiplication, division, and computing powers and roots on the magnitudes of quantities

9-12.1.3.1 develop fluency in operations with real numbers, vectors, and matrices, using mental computation or paper-and-pencil calculations for simple cases and technology for more-complicated cases

9-12.1.3.2 judge the reasonableness of numerical computations and their results

9-12.2.1.2 understand relations and functions and select, convert flexibly among, and use various representations for them

9-12.2.1.4 understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, using technology to perform such operations on more-complicated symbolic expressions

9-12.2.1.6 interpret representations of functions of two variables

9-12.2.2.1 understand the meaning of equivalent forms of expressions, equations, inequalities, and relations

9-12.2.2.2 write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency—mentally or with paper and pencil in simple cases and using technology in all cases

9-12.2.2.5 judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology

9-12.3.3.2 use various representations to help understand the effects of simple transformations and their compositions

9-12.6.1 build new mathematical knowledge through problem solving

9-12.6.2 solve problems that arise in mathematics and in other contexts

9-12.6.3 apply and adapt a variety of appropriate strategies to solve problems

9-12.6.4 monitor and reflect on the process of mathematical problem solving

9-12.7.2 make and investigate mathematical conjectures

9-12.8.1 organize and consolidate their mathematical thinking through communication

9-12.8.2 communicate their mathematical thinking coherently and clearly to peers, teachers, and others

9-12.8.3 analyze and evaluate the mathematical thinking and strategies of others

9-12.9.1 recognize and use connections among mathematical ideas

9-12.9.2 understand how mathematical ideas interconnect and build on one another to produce a coherent whole

9-12.10.1 create and use representations to organize, record, and communicate mathematical ideas

9-12.10.2 select, apply, and translate among mathematical representations to solve problems



Question 1:

Provide an equation containing at least one radical function where the algebraic solution yields one real solution and two extraneous solutions.

There are an infinite number of solutions to this problem.

Explain your reasoning.

Question 2:

Without algebraically solving this problem, predict how many real and extraneous solutions would be obtained when proceeding to solve the following radical equation algebraically.

$$\sqrt{x+3} - 2 = 2x + 3$$

Multiple responses exist. Answers will vary.

Explain your reasoning.

Below, you see two different (basic) radical equations solved algebraically using the functionality of the CAS. Examine each, discuss both solutions in your assigned group. Record your responses and notes on the lesson worksheet in your groups.

What do you notice differently about each solution?

Students should recognize that the variable x only appears once in figure 1. Squaring both sides does not increase the power of the equation to a quadratic. However, advanced students may get ahead and be thinking $(\sqrt{x})^2 = \sqrt{(x^2)} = |x|$. There are some detailed mathematical notes concerning this equality on page four of the teacher notes page.

Would you have solved them differently? Why or why not? Explain.

We hope that our classroom of students communicate well. Here, some students in figure 2 may set the equation to zero and solve. This is not incorrect, it is just not efficient.

Is there anything missing? Explain why or why not.

Figure 2 fails to check and discover x=-1 is an extraneous solution, but that is ok at this point. It is also possible students see the |x| issue previously discussed.

$\sqrt{x-2} = 5$	$\sqrt{x-2} = 5$	$\sqrt{x+2} = x$	$\sqrt{x+2} = x$
$(\sqrt{x-2} = 5)^2$	$x-2=25$	$(\sqrt{x+2} = x)^2$	$x+2=x^2$
$(x-2=25)+2$	$x=27$	$(x+2=x^2)-x-2$	$0=x^2-x-2$
		factor (x^2-x-2)	$(x-2) \cdot (x+1)$
		solve $((x-2) \cdot (x+1)=0,x)$	$x=-1$ or $x=2$
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Figure 1.

Figure 2.



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After recording your responses to these questions, proceed to open the file on the TI-Nspire CAS handheld file titled *ExtraneousSolutions.tns*, read the introduction on page 1.2. Complete page 1.3 and then check your understanding with a question on page 1.4.

It is ok if you are not 100% correct, we will examine further.

Here at this point, students are not expected to be “checking” their answers for radical equations. This is the construct to have them discover “WHY”. We traditionally have told students to do this. Page 1.3 may introduce questions from students on “Operation might introduce false solutions”. This is OK. It begins examination on page 1.5 directly and again in problem 2. Page 1.4 may not end up with students selecting “extraneous solution” for an answer. OK!

Proceed to page 1.5 and answer this question. What do you notice at the bottom of the CAS screen after you complete the first algebraic step in solving the problem on page 1.5?

Discuss why this warning “pops up” in your group. Record your discussions.

Problem 1 summary:

Throughout problem 1, the focus was almost entirely skill-based. However, a couple of things may/should have appeared.

If you caught them, great! On page 1.6, what do you notice at the bottom of the screen after you perform the next algebraic step required to solve the problem? Go ahead and begin solving the problem on page 1.6.

Why do you think this statement 'pops up' when performing the second line procedure?

Record your group's responses for discussion.

In the summary of problem 1, we hope that some students begin to connect squaring a square root to the $|x|$ issue previously discussed, though they may not be able to put it into words yet.

PROBLEM 2:

Examine the problem below. You'll see both the algebraic and graphical solutions to one of the equations examined in problem 1.

Discuss in your groups, what do you see "graphically" that may not have been discovered during the completion of problem 1.

Record your thoughts on the worksheet as to why each of the two methods do not yield identical solutions to the equation $\sqrt{x+2} = x$

We hope that students begin to think what squaring an equation does graphically to the solution, but OK if not there yet entirely.

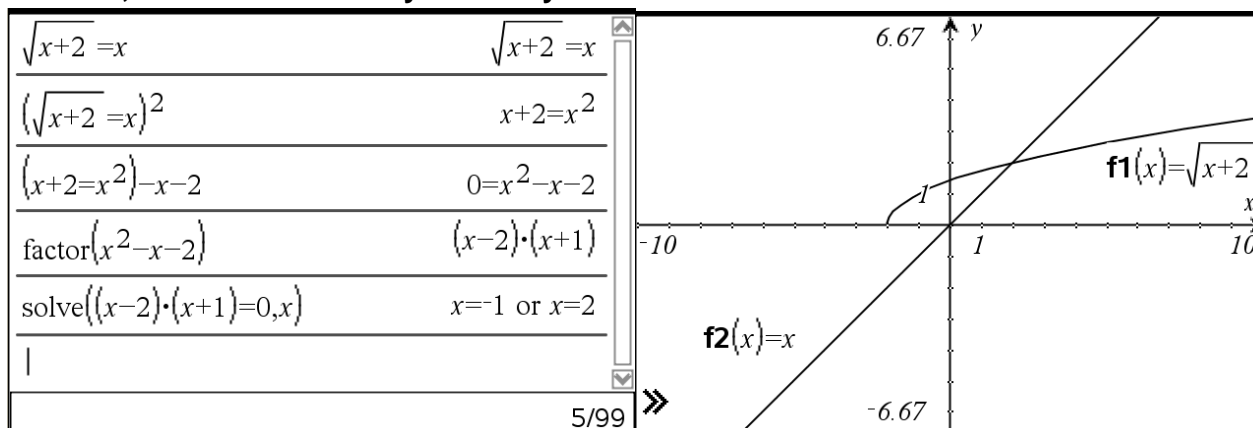


Figure 3.

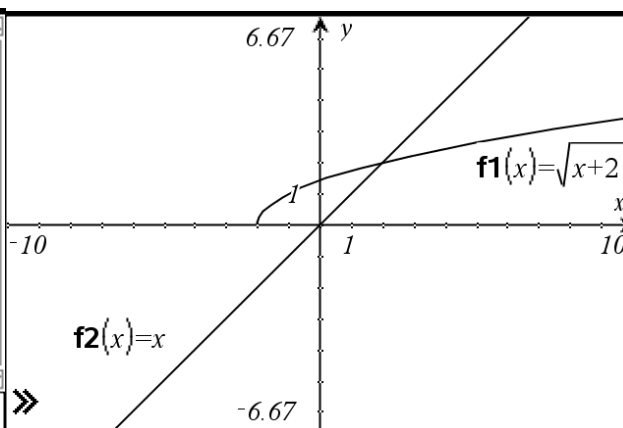


Figure 4.

Above, you see the algebraic solution has two solutions and a graphical equation shows only one solution. In problem 1, you made the discovery that squaring both sides of the equation introduces the possibility of false (extraneous) solutions. Let's examine a "math fact" that we often forget or take for granted. That is, $(\sqrt{x})^2 = \sqrt{(x^2)} = |x|$. **Note: Here is where the issue arises for discovery. The first part of the math fact has domain $x \geq 0$ while the second part is all real. This is a wonderful chance to see the importance of the domain concept that can be lost when algebraic solutions are the primary focus and domain is just something from earlier in the term. This problem provides a rich context of problem solving when domain should not be ignored. Conversations should take place during or near the end of the first day of the lesson regarding the domain concept, especially for the square root function.**

Proceed to page 2.2, you see $f_1(x)$ and $f_2(x)$ as in figure 4. Using the math fact above, graph the 'other' part of $f_1(x)$ so that the extraneous solution $x = -1$ appears graphically. Then, proceed to page 2.3 and check your understanding. Discuss this in your group until everyone understands. Record your group's concern, questions, and explanation.

RESPONSE:

Here is where we hope students begin to examine the +/- of the radical function and begin to make connections to the math fact listed above when squaring equations.



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Problem 2 summary:

You now "see" the possibilities of solution types (real & extraneous) to square root radical equations. So far, you've been working mostly with the square root of a linear function set equal to a linear function. For example, $\sqrt{x+3} = x+1$ was one of the first equations we saw in this lesson. By understanding the graphical representation of this equation, we can easily predict how many real solutions and extraneous solutions we will have if we proceed algebraically in solving the equation. You now know the "WHY" behind extraneous solutions from a graphical point of view that exist when solving square root equations algebraically.

In space provided, give a radical (square root) equation involving only linear functions (as we've seen) that would have no real solution and two extraneous?

At this point, students could have many different attempts or solutions. If they think graphically, horizontal lines below the x-axis and \sqrt{x} are simple examples.

Equation: _____

Discuss your equations in your group. Arrive at a consensus.

Give supporting evidence of your equation.

If you are still trying to conceptualize this lesson, indicate where you would like more explanation.

RESPONSE:

PROBLEM 3:

Proceed to problem 3 on the CAS handheld. Complete page 3.2 and 3.3 in your groups. Discuss and record any questions, comments, or concerns you group members may have with these two problems. Proceed to page 3.4. The class will then discuss the differences between square root and cube root radical equations.

Questions, comments, concerns:

At this point, we hope that students discover and conjecture that squaring equations may introduce extraneous solutions while cubing does not. Clarification is important here. This is a nice place to end a class with discussions. Transformations of radical and linear functions come next with Problem 4 and is a nice starting place to begin the final day.



PROBLEM 4:

Proceed to problem 4 on the CAS handheld. Read page 4.1.

Transformation instructions: The graph on page 4.2 has sliders inserted that allow for the parameters to be changed dynamically for the function $f_2(x) = a\sqrt{x+b}+c$. The linear function $f_1(x) = x$ can have the slope and y-intercept modified by grabbing the line near the end (circled arrows symbol appears) or grabbing near the y-intercept (cross-hair symbol appears), respectively. These two features allow both functions to be transformed.

On page 4.2, see which different types of solution sets you can obtain for the radical equation

$$a\sqrt{x+b}+c = mx+d$$

Record and discuss the possibilities in your group. Then, proceed to check your understanding on page 4.3.

RESPONSE:

We hope that students discover that transforming the composition function (square root of a linear) and transforming a linear function results in only a few possibilities for solution sets to radical equations of this form. This is what will set students up for extending beyond this typical type of equation that involves more complex types of functions in the equations.

PROBLEM 5:

Proceed to problem 5 on page 5.1. Members within your group should attempt to complete page 5.1 differently. That is, do different steps to solve the problem. Record the differences in solutions you may obtain. Discuss and then proceed to page 5.2, read the problem and then use page 5.3 to check your understanding on page 5.2. Then, move to page 5.4. Record response below. After completing page 5.5, discuss why you answered page 5.2 as you did. Clarify your correct or incorrect response.

RESPONSE:

Note: 5.2 is a loaded question. There is a solution out beyond $x=500$, so the point is that students need to “think” of all possibilities than limiting their thought to just the 10×10 grid of the Cartesian Plane.

5.5 begins to incorporate more complex equations and allows for more discovery with the rigorous paper and pencil manipulation required to solve algebraically. But, students should not rely solely on graphical solutions only without deep thought of the functions that make up the equation.



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For homework, now answer questions 1 and 2. Provide written explanations, algebraic and graphical solutions that support your responses. We will discuss next time.

Question 1:

Provide an equation containing at least one radical function where the algebraic solution yields one real solution and two extraneous solutions.

There are an infinite number of equations that satisfy this problem.

Explain your reasoning.

Probably the simplest example students come up with is a quadratic with a negative leading coefficient with the axis of symmetry being the y-axis and a positive y-intercept and tangent to a radical function transformed (shifted) to the left. NOTE: $\sqrt{x+4} = -x^2 + 2$ is a typical response from students. However, careful zooming near the y-intercept indicates TWO real solutions. Therefore, students have to be careful to make sure the quadratic and radical function are tangent. Another possible solution is a simple radical and a cubic transformed so that a point of tangency is the real solution OR that a point of tangency exists as an extraneous solution. Students can also come up with a trig function and a radical function. Many possibilities exist if you extend the possible function types beyond just polynomials.

Question 2:

Without working out this problem with paper and pencil, predict how many real and extraneous solutions would be obtained when proceeding to solve the following radical equation algebraically.

$$\sqrt{x+3} - 2 = 2x + 3$$

Multiple responses exist. Answers will vary.

Explain your reasoning.

Here, typically the efficient solution involves adding 2 to the equation first. This will then result in one real and one extraneous solution algebraically. This can be visualized. However, by squaring both sides first, a second squaring is required. This actually results in one real and three extraneous solutions to the problem because a quartic equation results. This is a great opportunity to have students explore the 'four solution' possibility graphically and algebraically with technology where tedious algebraic manipulations are not the focus.