EXPLORATIONS

**A C T I V I T Y   7**

Evaluating Rational Functions

A rational function is a function that can be expressed as $f(x) = \frac{h(x)}{g(x)}$, where $h(x)$ and $g(x)$ are both polynomials and $g(x)$ is not the zero polynomial. For this activity, it will be assumed that $h(x)$ and $g(x)$ have no common factors.

Some rational functions exhibit two key characteristics—*vertical asymptotes* and *horizontal asymptotes*—in addition to zeros and *y*-intercepts. Others do not have either.
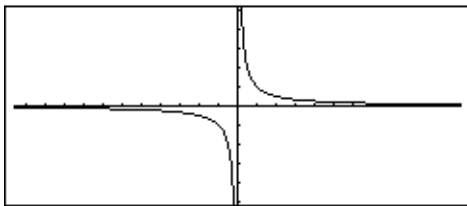
The asymptotic behavior of rational functions that contain both types of asymptotes is easily observed in the graph of $f(x) = \frac{1}{x}$.

Notice that, as the values of *x* increase without bound towards positive and negative infinity, the *y* values tend to 0. This is an example of a horizontal asymptote.

As the values of *x* tend to 0, the values of *y* increase without bound to infinity for values to the right of zero and decrease without bound to negative infinity for values to the left of zero. This is an example of a vertical asymptote.

In the activity, you will explore the two types of asymptotes and make generalizations as to when they do or do not occur.

Evaluating Rational Functions *(Continued)*

Instructions

Part A—Finding the Asymptotes of a Function

Explore the behavior of the function $f(x) = \frac{2x^2 - 8}{x^2 - 16}$, using the Voyage™ 200 PLT.

1. Press ON to turn on the Voyage 200 PLT. To reset to the default settings, press 2nd MEM F1, select 1:RAM, select 2:Default, and then press ENTER ENTER.

    Press ♦ [CALC HOME] to access the Home screen. Clear all one-character variables by pressing F6, selecting 1:Clear a–z, and pressing ENTER.

2. To find the factors of the numerator and denominator, use the **factor(** command as follows:

    Press F2 and select 2:factor(. To enter the expression, press (**2** × **X** ^ **2** − **8**) ÷ (**X** ^ **2** − **16**) , **X**).

    Press ENTER to return the factorization of the function.

    Record the factorization below.

    f(x) = _____

3. From the factors, determine the zeros of the numerator and denominator by setting each factor equal to 0 and solving for x.

    Write the zeros of the numerator and denominator below.

    zeros of the numerator = _____

    zeros of the denominator = _____

4. To find the y-intercept of the function, find the value of the function at 0, using the symbol [|]. The "with" operator feature on the Voyage 200, [|] (accessed by pressing 2nd **K**), is displayed as a "bar" on the screen. It substitutes the expression that follows into the expression that precedes the "bar."

    Press (**2** × **X** ^ **2** − **8**) ÷ (**X** ^ **2** − **16**) 2nd [|] **X** = **0** ENTER.

    The value of the y-intercept is returned.

    Record the value of the y-intercept below.

    y-intercept = _____

Evaluating Rational Functions *(Continued)*

5. To determine the behavior of $f(x)$ near the zeros of the denominator, use the **limit(** command. The arguments for this command are the *expression, the variable, the value of x to be examined*, and *either a positive or negative one*. The last argument allows you to investigate the y-value to the right (positive value) or to the left (negative value) of the target value.

   (a) Press ⌈F3⌉ and select 3:limit(. Press
      ⌈(⌉ **2** ⌈×⌉ **X** ⌈^⌉ **2** ⌈−⌉ **8** ⌈)⌉ ⌈÷⌉ ⌈(⌉ **X** ⌈^⌉ **2** ⌈−⌉ **16** ⌈)⌉ ⌈,⌉ **X** ⌈,⌉ **4** ⌈,⌉ **1** ⌈)⌉ ⌈ENTER⌉.

      The value of y is computed as the values of x approach 4 from the "right" side of 4.

      To compute the value of y as the values of x approach 4 from the "left" side of 4, press ⊙ and ⊙ ⊙ to backspace until the cursor is in front of the 1. Then insert a ⌈(-)⌉. Press ⌈ENTER⌉ to return the y-value as the values of x approach 4 from the "left" side of 4.

      Describe below the behavior of $f(x)$ near $x = 4$. How does $f(x)$ increase or decrease near $x = 4$?

      _____

      _____

      _____

   (b) Repeat the procedure for −4 by editing the entry line with ⊙ and ⊙ to change the value of x to −4. Evaluate the limit with the last argument as both 1 and −1.

      Describe below the behavior of $f(x)$ near $x = -4$. How does $f(x)$ increase or decrease near $x = -4$?

      _____

      _____

      _____

   (c) If the values of $f(x)$ increase or decrease without bound near an x-value, such as a, then $x = a$ is a *vertical asymptote*.

      Thus, the vertical asymptotes of $f(x)$ are

      $x =$ _____ and $x =$ _____ .

Evaluating Rational Functions *(Continued)*

6. The behavior of $f(x)$ as x increases to positive and negative infinity can be found using the **limit(** command with ∞ and then -∞ as the values of x. You can do this by editing the limit expression on the entry line as follows.

    Press ⊙ and then ← to delete the last two arguments. Press 2nd [∞] for the infinity symbol, and then press) to close the expression. The last argument is not needed. Press ENTER to return the value.

    Repeat for x decreasing without bound by editing the expression on the entry line so that the value of x is -∞.

    To what value does $f(x)$ approach in both directions? Write your answer below.

    $f(x)$ *approaches* = _____

    If the value of $f(x)$ approaches a constant value, such as c, as x increases or decreases without bound, then $y = c$ is a *horizontal asymptote*.

    Thus, the horizontal asymptote of $f(x)$ is

    $y =$ _____ .

7. To graph $f(x)$ on the Voyage™ 200 PLT, press ◆ [Y=] to display the Y= Editor. Press F1, select 8:Clear Functions, and press ENTER. Press F5, and select 5: Data Plots Off to turn off all data plots.
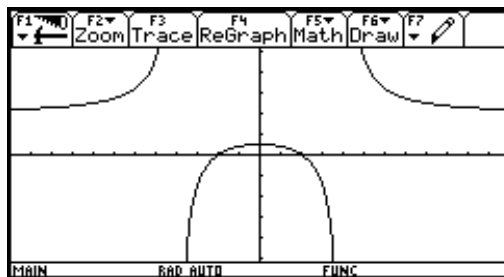
    Press ENTER to move the cursor to the entry line. Press
    (2 × X ^ 2 − 8) ÷ (X ^ 2 − 16) ENTER.

    The function $f(x)$ is now defined as y1.

    Press ◆ [WINDOW], and then press ⊙ to highlight the value of xres. Press **1** to change the resolution to the greatest detail.

    Press F2, and select 4:ZoomDec to produce the graph shown to the right.

Evaluating Rational Functions *(Continued)*

Part B—Solving Additional Problems

For each rational function given on the following pages, follow these directions.

a. Press ◆ [CALC HOME] to return to the Home screen. Clear the entry line on the Home screen by pressing CLEAR. Clear the Home screen by pressing F1 and selecting 8:Clear Home.

b. Find the factors of the numerator and denominator by using the **factor(** command. Express $f(x)$ in factored form, and record in the space provided.

c. From the factors, determine the zeros of each factor. Record the values in the table.

d. Find the y-intercept of the function by evaluating it at 0. Record the value in the table.

e. Use the **limit(** command to determine the behavior of $f(x)$ as x approaches each zero of the denominator for values to the right and left of the zero. Record the value(s) in the table.
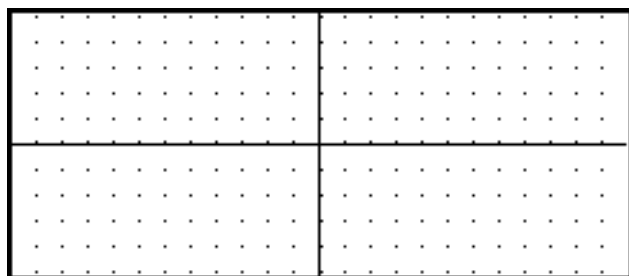
From this information, write the vertical asymptote(s) in the table.

f. Use the **limit(** command to determine the behavior of $f(x)$ as x approaches both positive and negative infinity. Record the value(s) in the table.

From this information, write the horizontal asymptote in the table.

g. Sketch the graph based on the information on the grid provided. Label the zeros and y-intercept, and sketch both the vertical asymptote(s) and the horizontal asymptote. Confirm the graph by graphing the function on the Voyage™ 200 PLT.
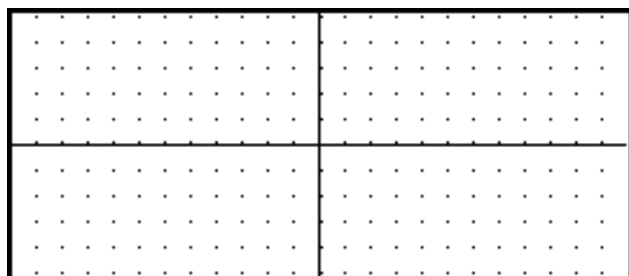
Evaluating Rational Functions *(Continued)*

1. $f(x) = \dfrac{4x-4}{2x+2}$ Factored form of $f(x)$ =

Zeros of the numerator		Zeros of the denominator		y-intercept	
Behavior of $f(x)$ near the zeros of the denominator			Vertical asymptote(s)		
Behavior of $f(x)$ as x approaches ±∞			Horizontal asymptote		

2. $f(x) = \dfrac{5x-10}{x^2-3x-10}$ Factored form of $f(x)$ =

Zeros of the numerator		Zeros of the denominator		y-intercept	
Behavior of $f(x)$ near the zeros of the denominator			Vertical asymptote(s)		
Behavior of $f(x)$ as x approaches ±∞			Horizontal asymptote		

Evaluating Rational Functions *(Continued)*

3. $f(x) = \dfrac{-2x^2 + 8}{x^2 + 4}$ Factored form of $f(x)$ =

Zeros of the numerator		Zeros of the denominator		y-intercept	
Behavior of $f(x)$ near the zeros of the denominator				Vertical asymptote(s)	
Behavior of $f(x)$ as x approaches $\pm\infty$				Horizontal asymptote	
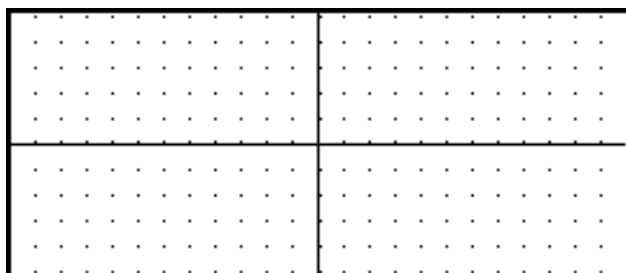
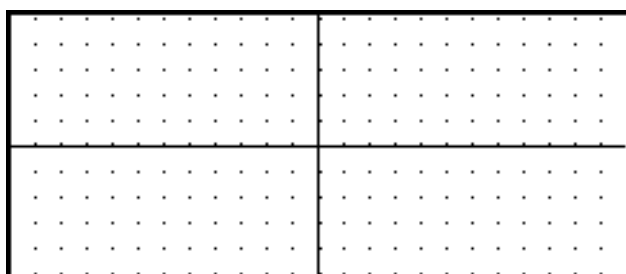4. $f(x) = \dfrac{x^2 - 2x - 3}{3x + 6}$ Factored form of $f(x)$ =

Zeros of the numerator		Zeros of the denominator		y-intercept	
Behavior of $f(x)$ near the zeros of the denominator				Vertical asymptote(s)	
Behavior of $f(x)$ as x approaches $\pm\infty$				Horizontal asymptote	

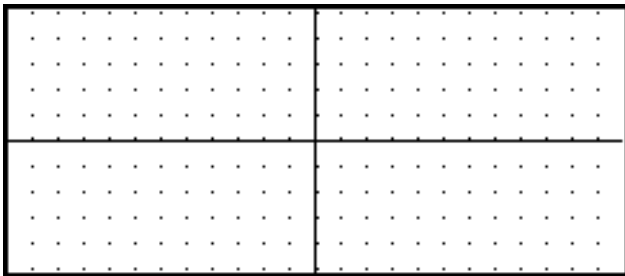## Evaluating Rational Functions *(Continued)*

5. $f(x) = \dfrac{3x^2 - 9}{x^3 - 2x^2 - 24x}$ Factored form of *f(x)* =

Zeros of the numerator		Zeros of the denominator		*y*-intercept	
Behavior of $f(x)$ near the zeros of the denominator			Vertical asymptote(s)		
Behavior of $f(x)$ as x approaches $\pm\infty$			Horizontal asymptote		

Evaluating Rational Functions *(Continued)*

Questions

1. For each of the rational functions in *Part B*, which graphs have vertical asymptotes? Why? What is the behavior of those graphs near the zeros of the denominator?

   _____

   _____

2. For each of the rational functions in *Part B* that had $y = 0$ as a horizontal asymptote, how do the degrees of the numerator and denominator compare?

   _____

   _____

3. For each of the rational functions in *Part B* that did not have a horizontal asymptote, how do the degrees of the numerator and denominator compare?

   _____

   _____

4. For each of the rational functions in *Part B* that had $y = a$ $(a \neq 0)$ as a horizontal asymptote, how do the degrees of the numerator and denominator compare?

   _____

   Determine the ratio of the leading coefficients of the numerator and denominator. How does that value compare with the value of the horizontal asymptote?

   _____

   _____

5. From step 2 through step 4, three cases arise from comparing the degrees of the numerator and the denominator. Each of these three cases determines the horizontal asymptote of a rational function if it has one.

   If the degree of the denominator is greater than the degree of the numerator, then the horizontal asymptote is _____ .

   If the degree of the numerator is greater than the degree of the denominator, then the horizontal asymptote is _____ .

   If the degree of the numerator and the degree of the denominator are equal, then the horizontal asymptote is _____ .

   Hint: Look at the leading coefficients of the numerator and the denominator.

Evaluating Rational Functions *(Continued)*

Extensions

As you discovered, rational functions for which the degree of the numerator is greater than the degree of the denominator do not have horizontal asymptotes. For this particular case, you can examine the end behavior of the function as x increases or decreases without bound by using the **propFrac(** command.

This command is found in the Algebra ($\boxed{F2}$) menu. It returns the rational function $f(x) = \dfrac{h(x)}{g(x)}$ in the proper form $f(x) = \dfrac{r(x)}{g(x)} + q(x)$. The degree of $r(x)$, the remainder, will always be smaller than $g(x)$, the divisor. Therefore, as x increases or decreases without bound, the term $\dfrac{r(x)}{g(x)}$ will approach zero, leaving $q(x)$ as the end behavior model.

This technique also works with horizontal asymptotes. As an example, from the cleared entry line on the Home screen, press $\boxed{F2}$, select 7:propFrac(, and type

$\boxed{(}$ **2** $\boxed{\times}$ **X** $\boxed{\wedge}$ **3** $\boxed{+}$ **6** $\boxed{\times}$ **X** $\boxed{-}$ **4** $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ **X** $\boxed{-}$ **2** $\boxed{)}$ $\boxed{,}$ **X** $\boxed{)}$

$\boxed{ENTER}$. The function is returned as $\dfrac{24}{x-2} + 2 \cdot x^2 + 4 \cdot x + 14$. As x increases or decreases without bound, $\dfrac{24}{x-2}$ approaches zero, leaving $2 \cdot x^2 + 4 \cdot x + 14$ as the end behavior model.

Investigate the behavior of each of the following rational functions by determining the zeros, y-intercept, vertical asymptote(s), and end behavior model.

Graph the function in y1 and its end behavior model in y2. To adjust the window to see the complete graph, press $\boxed{\bullet}$ $\boxed{WINDOW}$, highlight the value you wish to change, type the new value, and press $\boxed{ENTER}$.

Press $\boxed{\bullet}$ $\boxed{GRAPH}$ to display your new graph. On a separate sheet of paper, summarize your findings in your own words.

1. $f(x) = \dfrac{x^2 - 2}{x}$

2. $f(x) = \dfrac{x^3 - 2x}{x + 2}$

3. $f(x) = \dfrac{-x^4 + 3x^2 + 4}{x^2 - 16}$