# Ages 15-17 – Trivialisation and dynamisation with examples from elementary vector geometry

*Classical advanced-level exercises are often considered difficult because their solution normally requires extensive calculations. The effort to develop suitable exercises, which can be solved in a reasonable amount of time, is usually considerable. Two examples from elementary vector geometry (distance calculation and parametric representation of straight lines) will illustrate how classic exercises can be trivialised using CAS.*

*Furthermore, these examples demonstrate that such exercises can be extended to become dynamic and more demanding tasks by parameterisation. For each example some similar exercises are given as activities for students. These can be used both as examination questions and for training purposes.*

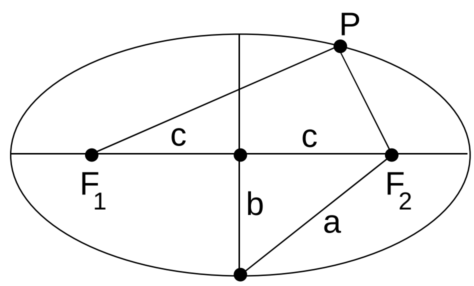
*The examples show that mathematics does not become trivial even though some exercises may be trivialised. Rewriting examples to include parameters means that students are required to know even more mathematics than they would need in a non-CAS situation.*

## 1. Example 1: Construction of an ellipse with a thread (category C1)

<u>Definition</u>

The ellipse is the locus of all points $P$, where the sum $s = 2a$ of the distances from $P$ to two fixed points $F_1$ and $F_2$ (focus) is a constant (length of the thread).

Let $2c = |F_1F_2|$ be the distance of the two foci (see figure), then $a$ is the semi-major axis of the ellipse and $b = \sqrt{a^2 - c^2}$ is the semi-minor axis.




a) Use Cabri Geometry to draw the ellipse with the above definition. The basic objects are the foci and the length of the string $s = 2a$.

b) Determine the equation of the ellipse. Choose foci $F_1 = (-c;0)$, $F_2 = (c;0)$ and the point on the ellipse as $P = (x, y)$. Set up the condition in accordance with the above definition. Transform the ellipse equation into the following form (use expand to simplify and don't forget to replace $c$):
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

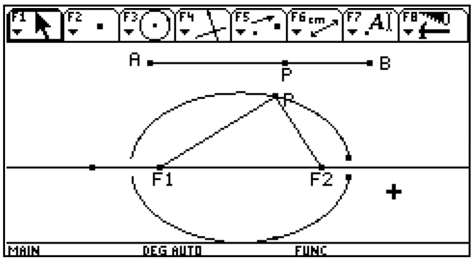
c) Solve the ellipse equation for $y$ and draw the two functions for $a = 4$ and $b = 3$.

d) Show that $x(t) = a\cos(t)$ and $y(t) = b\sin(t)$ satisfy the ellipse equation. Draw the ellipse in parametric representation for $a = 4$ and $b = 3$ and compare with c).

Solutions to example 1

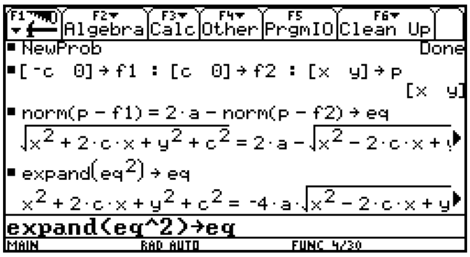
*First have students carry out this construction by-hand (working in pairs). To do this one merely needs a piece of thread (or string). One student holds the thread ends at the foci and the other student places the pencil at point P, drawing the ellipse by keeping the thread tight. The students can experiment freely or draw ellipses with given values for a and c.*

a) Possible solution: Construct a segment $\overline{AB}$ (length of the thread) and a point $P$ on it. Construct a circle with centre $F_1$ and radius $|AP|$ and another circle with centre $F_2$ and radius $|BP|$ (transmission of the segment). One of the intersection points is $P$. Draw the locus of both intersection points, while the corresponding point $P$ moves along the segment $\overline{AB}$.

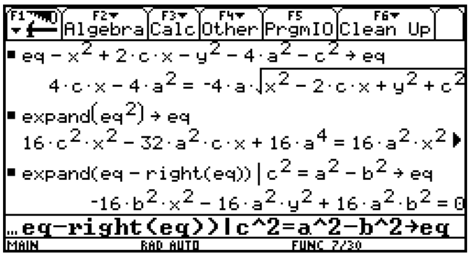



The ellipse will not be closed entirely, even if the maximum number of points is chosen. With the PC-version of Cabri you will obtain the result faster and the ellipse will be more beautifully formed.

b) Find a possible solution from the screenshot to the right. First you define the points. Then store the condition (equation *eq*) in such a way that by squaring it, a term as simple as possible arises. The next step: square and expand the equation and isolate the root on the right side.




Squaring once more and expanding delivers an equation without roots (the new equations are stored in `eq` at each step).

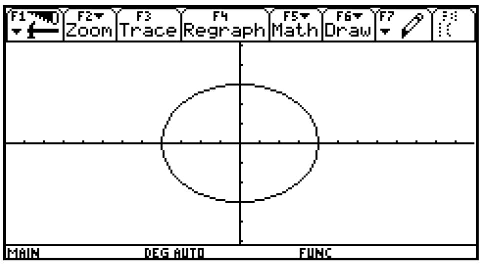



Using `eq-right(eq)` bring all terms to one side and in the same step substitute for $c^2$.
You will obtain the desired equation by dividing through by $-16a^2b^2$ and expanding.

The whole calculation can also be done by-hand. Use of the computer however shortens the computing time considerably.

c) Result: $y = \pm \dfrac{b\sqrt{a^2 - x^2}}{a}$.

Store the functions in $y1(x)$ and $y2(x)$ and draw the two halves of the ellipse. The ellipse is drawn completely if the points on the $x$-axis are ellipse points. You achieve this with `ZoomDec` and `xres=1` (window settings).



d) $\dfrac{x(t)^2}{a^2}+\dfrac{y(t)^2}{b^2}=\cos(t)^2+\sin(t)^2=1$

Switch function mode to parametric: MODE `Function/Parametric`, and angle to radian: MODE `Angle/Radian`. Draw the graph and you get the same figure as in c) (adapt window settings if necessary).

Note: Drawing on previous knowledge, you can also treat transformations in the coordinate system or use dilation e.g.: Translation by 3 in $x$-direction and 4 in $y$-direction.

$$\frac{(x-3)^2}{a^2}+\frac{(y-4)^2}{b^2}=1.$$

The ellipse arises from the unit circle $x^2+y^2=1$ by dilation in respect to the $x$- and $y$-axis with factors $a$ and $b$.

Another possibility is to introduce the equations of a hyperbola and parabola in an analogous way.

## 2. Exercises for example 1 - practise or examination, all of category C3

Store the ellipse equation as a function `el(x,y,a,b)`.

(1) Consider ellipses with centre $O = (0; 0)$ and semi-axes 5 and 3 on the x- and y-axis
 a) (classical) Does point (4; 2) lie on the ellipse?
 b) What are the conditions for $c$, so that point ($c$; 2) lies on (within) the ellipse?
 c) Determine $c$, so that point ($c$; 4) lies on the ellipse.
 d) Determine the semi-axes $a$ and $b$ and the foci of an ellipse, which goes through the points (4; 3) and (2; -5).

(2) Consider ellipses with centre $O$ and semi-axes 5 and 3 on the $x$- and $y$-axis.
 a) (classical) Find the intersection of the ellipse and the straight line $y=2x+1$.
 b) For which values of $c$ is $y=c\cdot x+1$ a tangent to the ellipse?
 c) For which values of $a$ is $y=x+c$ a tangent to the ellipse?
 d) Determine an ellipse with ratio 3:1 for the axes which touches the line $y=2x+1$ (centre $O$, axes of the ellipse on the $x$- and $y$-axis.
 e) For which values of $c$ does $y=c\cdot x+1$ cut a chord with length 8 of the given ellipse?

(3) Given: $A = (1; 0)$ and $B = (4; 0)$.
 a) (classical) For which points $P$ on the straight line $y = x + 1$ is the distance to point $B$ twice the distance to point $A$?
 b) Determine the locus of all points whose distance to point $B$ is twice the distance to point $A$. (equation and geometrical interpretation.)
 c) Complete the same task as for b), but this time let $B = (c; 0)$.
 d) Consider the same situation as in b) for the locus of all points with ratio $|BP| : |AP| = k$. What kind of locus is this? What are the intersection points of this locus with the $x$-axis? Where do these points tend to, if $k$ tends to 1? Interpret the result geometrically.

(4) Given: $A = (-12; -3)$, $B = (6; 9)$, $C = (13; 2)$

a) (classical) Determine the centre $M$ and the radius $r$ of a circle which goes through points $A$, $B$ and $C$.

b) Determine the equation of the circle found in a).

c) Solve task a) with $C = (13; k)$ and discuss the possible cases (geometrical interpretation).

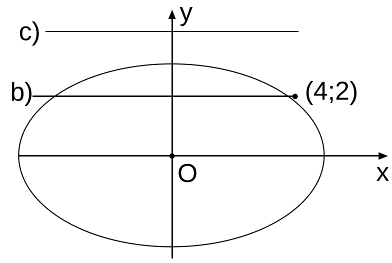
<u>Solutions to exercises 1 to 4</u>

S1 a) `el(4,2,5,3)` Result: false.

The point does not lie on the ellipse.

b) `solve(el(c,2,5,3),c)`

Result: $c = \pm\frac{5\sqrt{5}}{3}$ $\left(-\frac{5\sqrt{5}}{3} < c < \frac{5\sqrt{5}}{3}\right)$




c) `solve(el(c,4,5,3),c)` Result: false.

It may be that the major axis lies on the $y$-axis.

Therefore also test: `solve(el(c,4,3,5),c)`

Result: $c = \frac{9}{5}$ or $c = -\frac{9}{5}$.

d) `solve(el(4,3,a,b) and el(2,-5,a,b),{a,b})`

Result: $a = \frac{\sqrt{91}}{2} \sim 4.770, b = \frac{\sqrt{273}}{3} \sim 5.508$, i.e. the major axis points in the direction of the $y$-axis; therefore, $c$ can be calculated with $c = \sqrt{a^2 - b^2} = \frac{\sqrt{273}}{6} \sim 2.754$; this results in the foci $F_{1,2} = \left(0; \pm\frac{\sqrt{273}}{6}\right)$.

*The above tasks could also be solved by-hand, but the effort of solving one exercise is approximately as great as solving all the tasks with CAS. The main emphasis therefore shifts from calculating to analysing mathematical expressions.*

S2 a) `solve(el(x,2x+1,5,3),x)` Result: $P_1 = (0.971; 2.943), P_2 = (-1.889; -2.778)$.

b) The straight line intersects the $y$-axis at (0; 1), therefore you always have two intersection points. Thus a tangent is impossible. You can also solve this by calculating the intersection points as in a). The discriminant $25c^2+8$ is always positive.

c) `solve(el(x,x+c,5,3),x)` Result: $x_{1,2} = \frac{5\cdot(-5c \pm 3\sqrt{34 - c^2})}{34}$.

With the double solution, i.e. $c_{1,2} = \pm\sqrt{34}$, you get the tangent.

d) `solve(el(x,2x+1,a,a/3),x)` Result: $x = \frac{-18 \pm \sqrt{37a^2 - 9}}{37}$.

Hence the lengths of the semi-axes are $a = \frac{3\sqrt{37}}{37}$ and $b = \frac{\sqrt{37}}{37}$.

Don't forget the other solution with

`solve(el(x,2x+1,a/3,a),x)` Result: $x = \dfrac{-2 \pm \sqrt{13a^2 - 9}}{13}$

and we get an ellipse with $a = \dfrac{3\sqrt{13}}{13}$ and $b = \dfrac{3\sqrt{13}}{13}$.

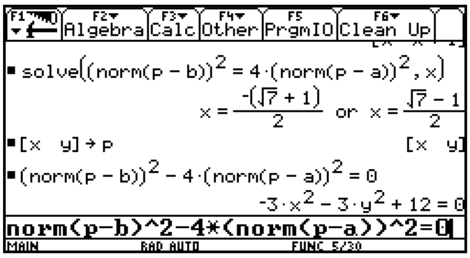
e) `solve(el(x,c*x+1,5.3),x)` Result: $x_{1,2} = \dfrac{5 \cdot (-5c \pm 3\sqrt{25c^2 + 8})}{25c^2 + 9}$

Difference of $x$-values: $x_2 - x_1 = \dfrac{30\sqrt{25c^2 + 8}}{25c^2 + 9}$ (store it in $d$).

Difference of $y$-values: $y_2 - y_1 = c \cdot (x_2 - x_1) = c \cdot d$.

Distance: `solve(d√(1+c^2)=8,c)` Result: $c_{1,2} = \pm 0.605$.

S3 a) First store the points $A$, $B$, $P$:
`[1,Ø]` STO▸ `a`, `[4,Ø]` STO▸ `b`, `[x,x+1]` STO▸ `p`.
See solutions in the screenshot to the right.




b) With $P = (x; y)$ you get the equation of the circle $x^2 + y^2 = 4$ (divide the last equation by -3), that is, the circle with centre at the origin and radius 2. This is the Apollonian circle over $\overline{AB}$ with ratio 2.

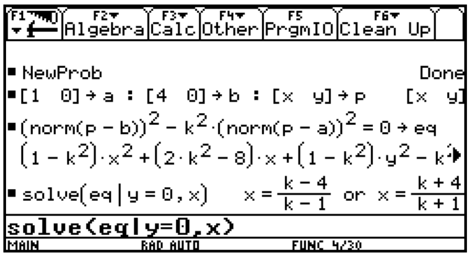
c) `[c,Ø]` STO▸ `b` and dissolution as in b) gives the equation $x^2 + \dfrac{2 \cdot (c-4) \cdot x}{3} + y^2 - \dfrac{c^2 - 4}{3} = 0$.

The equation can be transformed to standard form by completing the square. This only works by-hand, which shows that students still need to be able to perform simple algebraic manipulations by-hand: $\left(x - \dfrac{4-c}{3}\right)^2 + y^2 = \dfrac{4c^2 - 8c + 4}{9}$.

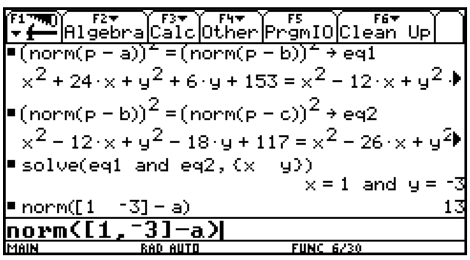
This is a circle with centre $M = \left(\dfrac{4-c}{3}; 0\right)$. The right side is the square of the radius.

For example with $c = 100$ the centre is (32; 0) and the radius is approximately 48.069.

d) The two solutions behave differently for $k \to 1$ (see accompanying screenshot). The second value tends to 2.5 (inner point to the ratio $k$), i.e. to the centre of the segment $\overline{AB}$. The first value tends to $\pm\infty$. This corresponds to a circle with an "infinitely" large diameter, i.e. the perpendicular bisector of $\overline{AB}$.




S4 a) First store the points $A$, $B$, $C$ and $P = (x; y)$. The conditions $|AP| = |BP|$ and $|BP| = |CP|$ correspond to the two perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$. Their intersection point is the centre $M = (1; -3)$ of the circle we found. The radius is $r = |MA| = 13$.

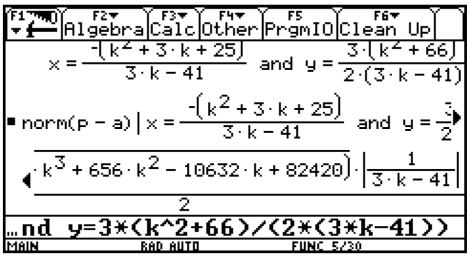


b) $|AP| = 13$

With CAS: `norm(p-a)^2=13^2` Result: $x^2 + 24x + y^2 + 6y + 153 = 169$.

Complete the squares to give: $(x + 12)^2 + (y + 3)^2 = 169 = 13^2$.

c) Replace $C$ by $(13; k)$ and carry out the same steps as in the screenshot of a).




Result: $x = \dfrac{-(k^2 + 3k + 25)}{3k - 41}$, $y = \dfrac{3 \cdot (k^2 + 66)}{2 \cdot (3k - 41)}$

If $k$ tends to $\dfrac{41}{3}$ then $x$, $y$ and $r$ tend to $\infty$

($r$ see screenshot).

Interpretation: $A$, $B$ and $C = \left(13; \dfrac{41}{3}\right)$ lie on a straight line.

## 3. Example 2: Elementary vector geometry in space (Category C3)

Given the points $A = (4; -5; 11)$, $B = (1; -\frac{1}{2}; 2)$ and the straight line $g$: $\vec{r} = \begin{pmatrix} 0 \\ -13 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$.

a) (classical) Determine all points $P$ on the straight line g with ratio $|AP| : |BP| = 2:1$.

b) Replace the third component 1 in the direction vector of the straight line $g$ with $c$ (in this case the direction of the straight line becomes variable). Which values of the parameter $c$ results in one, two or no solution?

c) On which locus do these intersection points lie? (points with $|AP| : |BP| = 2:1$, geometrical reasoning).

d) Determine the characteristic quantities of this locus.

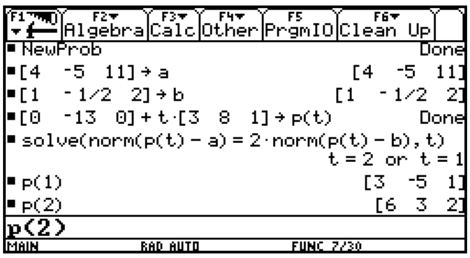
e) We now omit the condition $P \in g$, i.e. $P = (x; y; z)$. Determine the equation for $x$, $y$, $z$ of the locus in d). Show that it is the equation of a sphere ($|MP|^2 = r^2$) applying d).

Solutions to example 2

a) The problem can be described by the following conditions:

I. $\dfrac{\left|\overrightarrow{AP}\right|}{\left|\overrightarrow{BP}\right|} = 2$ or $\left|\overrightarrow{AP}\right| = 2 \cdot \left|\overrightarrow{BP}\right|$ II. $P \in g$

Solution (see screenshot): First store the points (= position vectors) in $a$ and $b$ and the straight line in $p(t)$. Thus equation II is taken into account. Solve equation I according to the screenshot.

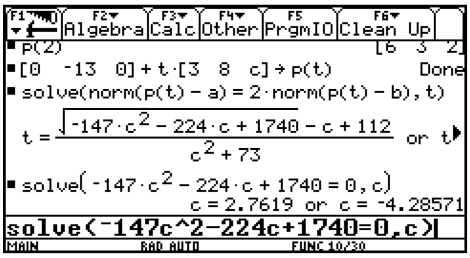



Result: $P_1 = (3; -5; 1)$ and $P_2 = (6; 3; 2)$.

*These simple results show that the exercise was prepared with much effort to reduce the amount of calculation by-hand. With CAS you can take any values without augmenting the calculation time. Thus it is much easier for the teacher to prepare exercises.*

*After setting up the conditions, the remaining task consists of rearranging the terms and solving a quadratic equation. CAS calculators trivialise this part regardless of the complexity of the result.*

b) With CAS you just change $p(t)$ and retrieve the equation from history. The equation was squared in order to accelerate calculations:




Result: $t_{1,2} = \dfrac{-c+112 \pm \sqrt{-147c^2 - 224c + 17}}{c^2 + 73}$ .

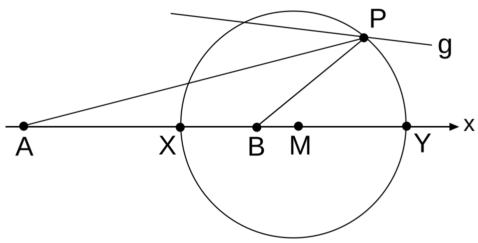
The solutions are determined by the discriminant:

- One solution: $c = -\dfrac{30}{7}$ = -4.286 and $c = \dfrac{58}{21}$ = 2.762,
- Two solutions: -4.286 < $c$ < 2.762,
- No solution: $c$ < -4.286 or $c$ > 2.762.

*With a single discussion of such a solution students will most likely learn more than by solving several drill exercises of type a).*

c) The points lie on the Apollonian sphere over the segment $\overline{AB}$ with ratio 2:1.

d) We determine the points $X$, $Y$ on the Apollonian sphere, which lie on the straight line $AB$ (diameter of the sphere).

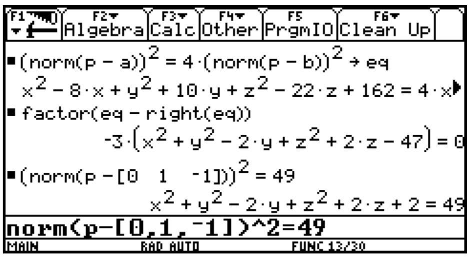



$\overrightarrow{OX} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$, $\overrightarrow{OY} = \overrightarrow{OB} + \overrightarrow{AB}$,

$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OX} + \overrightarrow{OY})$

Result: $X = (2; -2; 5)$, $Y = (-2; 4; -7)$,

e) `[x,y,z]` STO▸ `p`. For further steps see the screenshot on the right. Depending on interest and previous knowledge the equation can be brought to the standard form by completing the square:




$x^2 + (y - 1)^2 + (z + 1)^2 = 49 \Leftrightarrow |MP| = r^2$.

(Translation of the sphere $x^2 + y^2 + z^2 = 49$)

## 4. Exercises for example 2 - practise or examination, all of category 3

*The philosophy of all tasks is to replace numbers in example 2 by parameters and create new situations, which help to promote spatial imaginative power. The fact that CAS results are extensive forces a more abstract way of thinking which concentrates on the essentials of expressions.*

(5) Given: $A = (4; -5; 11)$, $B = (1; -\frac{1}{2}; 2)$ and the straight line $g$ (see example 1).

a) Let $c = 1$. Determine the points $P$ on the straight line $g$, for which $|AP| : |BP| = 2:1$.
b) Let $c$ be arbitrary. For which values of the parameter $c$ do solutions exist?
c) Interpret the result of a) and b) and compare results with those obtained for example 2.

(6) Given: $A$ and $B$ as in exercise 5 and $g$: $\vec{r} = \begin{pmatrix} c \\ 1 \\ d \end{pmatrix} + t \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$.

a) Let $c = 0$ and $d = 0$. Determine the points on the straight line $g$, for which $|AP| : |BP| = 3:1$.
b) Determine the radius $r$ of the Apollonian sphere over $\overline{AB}$ with ratio 3:1
c) Let $c = 10$ and $d = 0$. Complete the same task as for a).
d) Let $c = 10$ and $d$ be arbitrary. Complete the same task as for a).
e) Interpret the results of b) and c).

(7) Consider $A$ and $B$ as in exercise 5 and the straight line $g$: $\vec{r} = \begin{pmatrix} 0 \\ -13 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$ (see example 2).

Let us analyse the points $P \in g$, for which $|AP| : |BP| = k:1$ ($k>0$).

a) Find a formula for $t$ where $k$ is arbitrary.
b) The formula fails for a certain $k$. Discuss why this is the case.
c) For which other values do solutions exist?
d) Determine the radius of the Apollonian sphere $r$ using the intersection points with $AB$.
What happens as $k$ tends to 1?

<u>Solutions to exercises 5 to 7</u>

S5 a) Result: $t_{1,2} = \frac{\pm 7\sqrt{74}}{74}$, $P_1 = (2.41; 7.510; -0.186)$, $P_2 = (-2.441; -5.510; -1.841)$

b) Result: $t_{1,2} = \frac{\pm 7}{\sqrt{c^2 + 73}}$, i.e. there are always solutions.

c) The solutions are always symmetrical to the starting point of $g$.
Interpretation: The starting point of $g$ is the centre of the Apollonian sphere (compare with example 2). Therefore, every value of $c$ results in two intersection points.

*To further improve students spatial imagination ask them whether there are other straight lines with two intersection points for every c (symmetric to the starting point or not).*
*You will have the general case if the starting point S of g lies within the Apollonian sphere and the symmetric case if, in addition, g is perpendicular to SM (M = centre of the sphere).*

S6 a) Compared to example 2 only the ratio of the distances changes.
Result: $t_1$ = -0.496 and $t_2$ = 0.367, $P_1$ = (-1.487; -2.965; -0.496) and
$P_2$ = (1.102; 3.938; 0.367).

b) In analogy with example 2d) we calculate the intersection points of the Apollonian sphere with *AB*:

$$\overrightarrow{OX} = \overrightarrow{OA} + \frac{3}{4}\overrightarrow{AB},$$

$$\overrightarrow{OY} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{AB},$$

$$r = \frac{\left|\overrightarrow{XY}\right|}{2} = \frac{63}{16} = 3.938\,.$$

c) Result: false. The straight line doesn't intersect the Apollonian circle.

d) Here we get a formula for *t*: $t = -0.115 \cdot (0.117 \cdot (d + 34.750) \pm \sqrt{-d^2 + 2.726 \cdot d - 58.503})$ but the discriminant is always negative (CAS doesn't give you this important hint), i.e. there is no solution.

Geometric Interpretation: The plane in which the straight line moves as *d* is changed doesn't intersect the Apollonian sphere.

S7 a) Replace the ratio 2 by *k*: `solve(norm(p(t)-a)^2=k^2*norm(p(t)-b)^2,t)` gives

$$t = \frac{6 \cdot (35 \cdot k^2 - 29) \pm \sqrt{-6 \cdot (605 \cdot k^4 - 5691 \cdot k^2 + 4870)}}{148 \cdot (k^2 - 1)}\,.$$

b) For $k = 1$ you get a division by 0. If you solve the above equation with $k = 1$ the result is $t = -\frac{53}{48}$. Geometric interpretation:

The perpendicular bisector of $\overline{AB}$ intersects *g* at a point.

c) The discriminant in a) determines the solutions. They lie in the range $0.976 \le k \le 2.908$.

d) First store an arbitrary position vector `[x,y,z]` $\boxed{\text{STO}\blacktriangleright}$ `p` and the condition |AP| : |BP| = k:1 (Apollonian sphere) in the form `norm(p-a)^2=k^2*norm(p-b)^2` $\boxed{\text{STO}\blacktriangleright}$ `eq`.

Store the parametric equation of the straight line through *A* and *B* (the position vectors to *A* and *B* are still stored in *a* and *b*): `a+t*(b-a)` $\boxed{\text{STO}\blacktriangleright}$ `s(t)`.

Substitute *x*, *y* and *z* in *eq* with the components of *s*:

`eq|x=s(t)[1,1] and y=s(t)[1,2] and z=s(t)[1,3].`

Solving this equation with respect to t gives: $t_1 = \frac{k}{k+1}$ or $t_2 = \frac{k}{k-1}$ (not dependent on A and B!), whereas $t_1$ belongs to the inner intersection point *X* and $t_2$ to the outer one *Y*.

You get the centre of the Apollonian circle with $\frac{1}{2}(s(t_1)+s(t_2))$ or $s(\frac{1}{2}(t_1+t_2))$.

Result: $\mathrm{M}=\left(\frac{3}{2\cdot(k+1)}-\frac{3}{2\cdot(k-1)}+1;\frac{-9}{4\cdot(k+1)}+\frac{9}{4(k-1)}-\frac{1}{2};\frac{9}{2\cdot(k+1)}-\frac{9}{2\cdot(k-1)}+2\right)$

and the radius with $r=\frac{\left|\overline{XM}\right|}{2}$, result: $\frac{21}{2}\left|\frac{k}{(k-1)\cdot(k+1)}\right|$.

*Of course such a long formula has to be tested, e.g. with k → 0, 1, ∞, 3 (Exercise 6 b). Such discussions deal with limits in an intuitive way without any theory of limits.*

Note*:* A great number of classical exercises, which are trivialised by CAS, can be expanded to become demanding tasks by introducing parameters. Some of the examples above show that with such new tasks you will often come across subjects, which are usually dealt with later. I used this teaching sequence in the 10th school year as an application of vectors with coordinates.