Polynomial Approximations—An Introduction to Taylor Series

By Dave Slomer

Differentiable functions are so "locally linear" in a small enough "window" that the <u>tangent line</u> at a certain point $x = a$ quite well approximates the function's values for all x "near" enough to a. It stands to reason that a "tangent parabola" would be an even better approximation, allowing us to wander further from the point of tangency to get "good" approximations. A "tangent parabola" is one that is tangent to a given curve at a given point <u>*and*</u> which has the <u>*same concavity*</u> as the given curve at the point of tangency.

Exercise 0: All parabolic functions have the form $P(x) = ax^2 + bx + c$, where a, b, and c are constants (and a is not zero). Suppose that we want to approximate the function $y = f(x)$ with a tangent parabola, P. Write 3 general equations in terms of P and f that will make P tangent to f at $x = 0$. Functions P and f must <u>meet</u>, have the same <u>slope</u>, and have the same <u>concavity</u> at $x = 0$.

_____ = _____   _____ = _____   _____ = _____

Exercise 1: On your TI-89, **Define f(x) = e^(x)** and **Define p(x)=a*x^2+b*x+c**. Find the coefficients that will make **p** tangent to **f** at **x = 0**.

Give the command **DelVar a,b,c** before proceeding. Then give the commands listed below (these are the equations asked for in Exercise 0):

f(x)=p(x)|x=0
***d*(f(x),x)=*d*(p(x),x)|x=0**
***d*(f(x),x,2)=*d*(p(x),x,2)|x=0**

The "with" operator, "|" [located to the left of the $\boxed{7}$ key], enables making $x = 0$ in each of the 3 commands. Those commands will give you, at least indirectly, values for **a**, **b**, and **c**.

- What is the equation of the parabola that is tangent to $f(x) = e^x$ at $x = 0$?
- Graph both functions. Does your screen look like that in figure 1, where the window [-6,6] by [-1,5] was used?

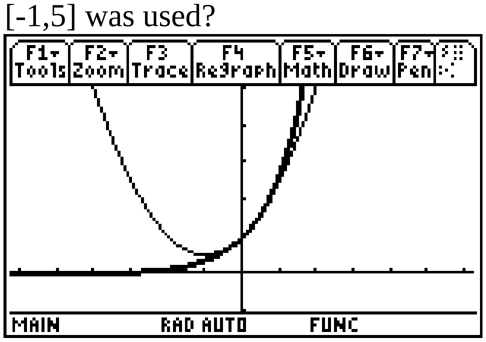

Fig. 1

- For which values of x does it <u>appear</u> that the parabola is a good approximation of the exponential?
- Trace to the smallest x in your previous answer and compare function values there. What's the difference?

- Trace to the largest x in your previous answer and compare function values there. What's the difference?
- To the nearest tenth, for which values of x will the parabola give results that are accurate (to tenths) approximations of the exponential? [You can get these values either by tracing or using your TI-89's table feature. If you trace, you might want to set **xres=1**.]

Exercise 2: Use the same basic procedure as outlined in Exercise 1 to find the line tangent to the exponential function at $x = 0$. (This is an "old" problem, revisited with this new twist.)

Define L(x)=m*x+n. Also, **Delvar m,n** before proceeding.

- What equation did you get for the tangent line?
- Graph it, along with the quadratic approximation (tangent parabola) and the exponential function. Does your graph screen look like that in figure 2?

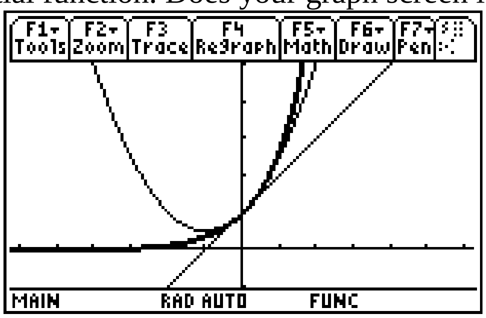

Fig. 2

Because the "**thick**" graph style was used for the exponential function in figure 2, it is hard to tell just how much better the parabola approximates the exponential than the line, but it is clearly better. (Convince yourself of that!)

- Did you notice that **p(x)=L(x)+**…*some other term…*? What is the other term?

Exercise 3: Find a cubic approximation for the exponential function.

Define r(x)=a*x^3+b*x^2+c*x+d.[**DelVar a,b,c,d** before proceeding] Since there are now four unknown constants, you'll need a fourth equation. So, equate the third derivatives of **r** and **f** at 0. Graph the cubic (**r**) along with the exponential (**f**) and quadratic (**p**). Does your graph look like that in figure 3? The window [-6,6] by [-5,25] was used with **xres=2** to give 3 distinct graph styles to help distinguish among the 3 graphs. Again, more [higher degree] is better [accuracy].

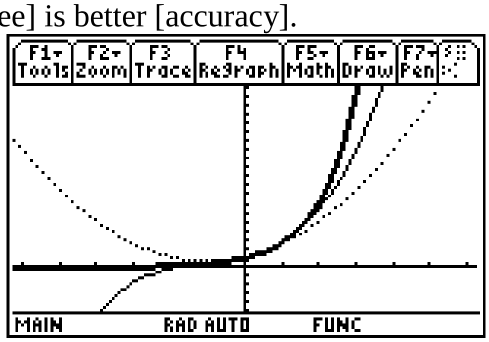

Fig. 3

- Did you notice that **r(x)=p(x)+**…*some other term*…? What is the other term?

Before proceeding, press [2nd][F6] to **Clear variables a-z**.

Exercise 4: Use the procedure from Exercises 0 through 3 to try to find a cubic approximation for $f(x) = \cos x$ at $x = 0$.

- Why do you think you can't find one? [TRY FIRST before answering!]
- Take whatever you <u>did</u> get and graph it along with the cosine function. In the **Zoom** [4] **(Decimal)** window, the graph screen will look the one in like figure 4. [Does the graph clearify why you can't find a cubic approximation?]

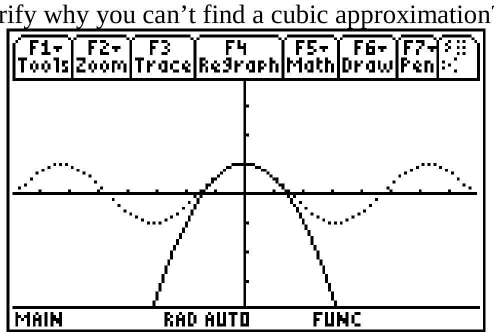

Fig. 4

- Without taking any derivatives, what do you think would be a good linear approximation for the cosine at $x = 0$?
- Graph that line along with the cosine and the other approximation. Does this linear function look like a decent approximation for the cosine near 0? [If not, you should use the procedure to find it.]
- Use your '89's table feature to see how accurate the two approximations are between -.4 and .4. Save some time by observing that both the cosine and its approximations are *even*.

Exercise 5: ***<u>Generalize</u>***.

Press [2nd][F6] to **Clear variables a-z** and then **Define p(x)=a*x^4+b*x^3+c*x^2+d*x+e** to be a *quartic* approximation of "generic" function **f**. Then find <u>formulas</u> for the coefficients of **p** by setting **p(0)=f(0)** and then setting the first four derivatives of **p** at 0 equal to the corresponding derivatives of **f** at 0. [Refer to figures 5 through 8, keeping in mind that the result equations returned by the '89 actually involve the derivatives <u>at **x = 0**</u>.]

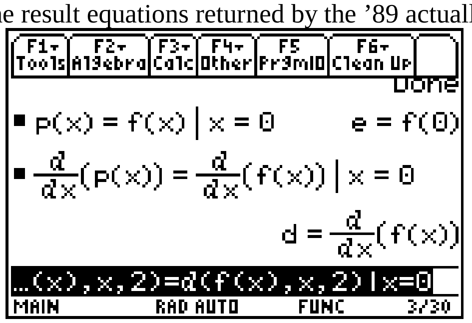

Fig. 5

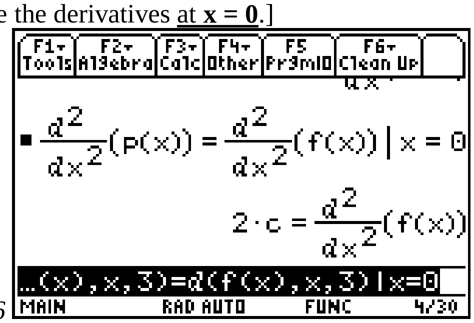

Fig. 6

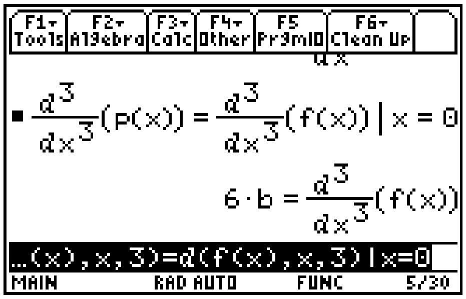

Fig. 7

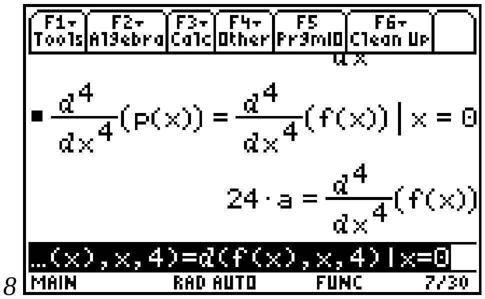

Fig. 8

The results of equating the function values and first 4 derivatives of **p** and **f** at 0 tell you, some directly, some indirectly, how to compute the coefficients **a**, **b**, **c**, **d**, and **e** for **p** to approximate **f**. Do a little algebra to fill in the table below, based on the results in figures 5 through 8.

Setting…	***…tells you that…***
f(0) = p(0)	the coefficient of the 0^{th} degree term, **e =**
f '(0) = p'(0)	the coefficient of the 1^{st} degree term, **d =**
f ''(0) = p''(0)	the coefficient of the 2^{nd} degree term, **c = f ''(0)/2**
f '''(0) = p'''(0)	the coefficient of the 3^{rd} degree term, **b =**
f ''''(0) = p''''(0)	the coefficient of the 4^{th} degree term, **a =**

You may be familiar with those coefficients in figures 5 through 8 and in the formulas in the table above. They are *factorials* of consecutive integers. In general, equating the n^{th} derivatives of **p** and **f** at 0 tells you that the coefficient of the n^{th} degree term in the polynomial approximation **p** is $\frac{f^{(n)}(0)}{n!}$, where $n! = 2 \cdot 3 \cdot 4 \cdots n$. Hence, in general, the n^{th} degree polynomial approximation of f is given by

$$p(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 \cdots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^n,$$

where $f^{(n)}(0)$ is taken to mean the n^{th} derivative of f at 0, $f^{(0)}(0)$ is taken to mean $f(0)$, and 0! is defined to be 1.

Ah, patterns…where would we be in mathematics without them and the ability to recognize them?

But it's a time-consuming procedure. However, there is good news! All of this is built into your TI-89! To get a 4^{th} degree polynomial approximation for f at 0, for example, give the command **Taylor(f(x),x,4,0)**. Refer to figure 9.

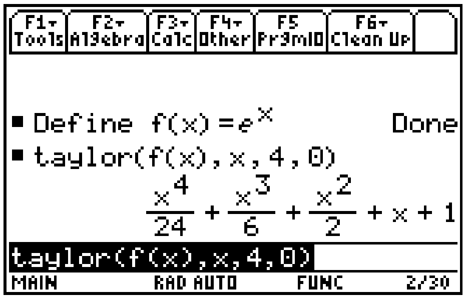

Fig. 9

Are the last 4 terms in figure 9 the cubic approximation that you found in Exercise 3?
Are the last 3 terms in figure 9 the quadratic approximation that you found in Exercise 1?
Are the last 4 terms in figure 9 the linear approximation that you found in Exercise 2?
What will be the <u>next</u> term in the Taylor polynomial if you get a 5th degree approximation?

Exercise 6: Find a 7th degree Taylor polynomial for the sine function at $x = 0$. Then graph both functions in the **Zoom** $\boxed{4}$ **(Decimal)** window. Compare. Conclusions? What degree is required to make the Taylor polynomial "look just like" the sine in that window? Define **y3(x)=y2(x)-y1(x)** and look at a table with **tblStart=0** and **Δtbl=1**. Page down through it. Describe the accuracy between 0 and 9.

Exercise 7: Find a 7th degree Taylor polynomial for the function $f(x) = e^{-1/x^2}$ at $x = 0$. Then graph both functions. Compare. Conclusions? (It may be instructive to compute the first few derivatives of f at 0, note that they are undefined, and then take their limits as x approaches 0. See figure 10. Conclusions??)

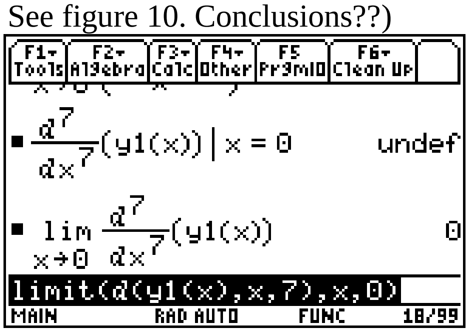

Fig. 10

But how accurate are the approximations? And how accurate do they need to be? And for which values of x do the approximations hold? These are questions that might be answered in a more advanced activity about Taylor series.

Calculus Generic Scope and Sequence Topics: Infinite Series, Applications of Derivatives
NCTM Standards: Number and operations, Algebra, Geometry, Measurement, Problem solving, Connections, Communication, Representation