Special Cases of the Product Rule

Student Activity – Teacher Answers

7 8 9 10 11 **12**

TI-Nspire™ | Activity | **Teacher** | 3 hours

Introduction

If $f(x)$ and $g(x)$ are both differentiable functions, then their product $f(x)g(x)$ is also differentiable, and using the product rule then:

$$(f(x)g(x))' = \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) = g(x)\frac{d}{dx}(f(x)) + f(x)\frac{d}{dx}(g(x))$$

In this activity we will meet examples of functions that also satisfy $(f(x)g(x))' = f'(x)g'(x)$ and examine and explore the patterns in the resulting differentials.

PART 1

Question: 1.

a) Consider the functions $f(x) = -(x+1)$ and $g(x) = \frac{1}{x}$, show that $(f(x)g(x))' = f'(x)g'(x)$

Answer

$f(x) = -(x+1), \quad g(x) = \frac{1}{x} = x^{-1}$

$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}\left(\frac{-(x+1)}{x}\right) = \frac{d}{dx}\left(-1 - x^{-1}\right) = \frac{1}{x^2}$

$f'(x)g'(x) = -1 \times \frac{-1}{x^2} = \frac{1}{x^2}$

Define $f(x)=-(x+1)$ ▸ *Done*

Define $g(x)=\frac{1}{x}$ ▸ *Done*

$\frac{d}{dx}(f(x)\cdot g(x))$ ▸ $\frac{1}{x^2}$

$\frac{d}{dx}(f(x))\cdot \frac{d}{dx}(g(x))$ ▸ $\frac{1}{x^2}$

b) Change $f(x)$ slightly so that $f(x) = x+1$ and check if it still satisfies: $(f(x)g(x))' = f'(x)g'(x)$

Answer: This example still works.

Define $f(x)=x+1$ ▸ *Done*

Define $g(x)=\frac{1}{x}$ ▸ *Done*

$\frac{d}{dx}(f(x)\cdot g(x))$ ▸ $\frac{-1}{x^2}$

$\frac{d}{dx}(f(x))\cdot \frac{d}{dx}(g(x))$ ▸ $\frac{-1}{x^2}$

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Question: 2.

a) Consider the functions $f(x)=(x+2)^2$ and $g(x)=\frac{1}{x^2}$, show that $(f(x)g(x))' = f'(x)g'(x)$

Answer:

$f(x)=(x+2)^2, \quad g(x)=\frac{1}{x^2}=x^{-2}$

$$\frac{d}{dx}(f(x)g(x))=\frac{d}{dx}\left(\frac{(x+2)^2}{x^2}\right)=\frac{d}{dx}\left(\frac{x^2+4x+4}{x^2}\right)$$

$$=\frac{d}{dx}(1+4x^{-1}+4x^{-2})=-4x^{-2}-8x^{-3}$$

$$=\frac{-4(x+2)}{x^3}$$

$$f'(x)g'(x)=2(x+2)\times\frac{-2}{x^3}=\frac{-4(x+2)}{x^3}$$

Define $\mathbf{f}(x)=(x+2)^2$ ▸ *Done*

Define $\mathbf{g}(x)=\frac{1}{x^2}$ ▸ *Done*

$\frac{d}{dx}(\mathbf{f}(x)\cdot \mathbf{g}(x))$ ▸ $\frac{-4\cdot(x+2)}{x^3}$

$\frac{d}{dx}(\mathbf{f}(x))\cdot \frac{d}{dx}(\mathbf{g}(x))$ ▸ $\frac{-4\cdot(x+2)}{x^3}$

b) Change $f(x)$ slightly so that $f(x)=x+2$ and check if it still satisfies: $(f(x)g(x))' = f'(x)g'(x)$

Answer: This one doesn't work:

Define $\mathbf{f}(x)=x+2$ ▸ *Done*

Define $\mathbf{g}(x)=\frac{1}{x^2}$ ▸ *Done*

$\frac{d}{dx}(\mathbf{f}(x)\cdot \mathbf{g}(x))$ ▸ $\frac{-(x+4)}{x^3}$

$\frac{d}{dx}(\mathbf{f}(x))\cdot \frac{d}{dx}(\mathbf{g}(x))$ ▸ $\frac{-2}{x^3}$

Question: 3.

a) Consider the functions $f(x)=-(x+3)^3$ and $g(x)=\frac{1}{x^3}$, show that $(f(x)g(x))' = f'(x)g'(x)$

Answer

$f(x)=-(x+3)^3, \quad g(x)=\frac{1}{x^3}=x^{-3}$

$$\frac{d}{dx}(f(x)g(x))=\frac{d}{dx}\left(\frac{-(x+3)^3}{x^3}\right)$$

$$=\frac{d}{dx}\left(\frac{-x^3-9x^2-27x-27}{x^3}\right)$$

$$=\frac{d}{dx}(-1-9x^{-1}-27x^{-2}-27x^{-3})$$

$$=9x^{-2}+54x^{-3}+81x^{-4}=\frac{9}{x^4}(x^2+6x+9)$$

$$=\frac{9(x+3)^2}{x^4}$$

$$f'(x)g'(x)=-3(x+3)^2\times\frac{-3}{x^4}=\frac{9(x+3)^2}{x^4}$$

Define $\mathbf{f}(x)=-(x+3)^3$ ▸ *Done*

Define $\mathbf{g}(x)=\frac{1}{x^3}$ ▸ *Done*

$\frac{d}{dx}(\mathbf{f}(x)\cdot \mathbf{g}(x))$ ▸ $\frac{9\cdot(x+3)^2}{x^4}$

$\frac{d}{dx}(\mathbf{f}(x))\cdot \frac{d}{dx}(\mathbf{g}(x))$ ▸ $\frac{9\cdot(x+3)^2}{x^4}$

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b) Change $f(x)$ slightly such that $f(x)=(x+a)^3$, determine the value for a such that it satisfies the condition:

$$\left(f(x)g(x)\right)'=f'(x)g'(x)$$

Answer: By substituting 'a' into f(x) and comparing the two derivatives, we can see that a = 3 is the only solution.

$\text{Define } f(x)=-(x+a)^3 \blacktriangleright \textit{Done}$

$\text{Define } g(x)=\frac{1}{x^3} \blacktriangleright \textit{Done}$

$\frac{d}{dx}(f(x)\cdot g(x)) \blacktriangleright \frac{3\cdot a\cdot (x+a)^2}{x^4}$

$\frac{d}{dx}(f(x))\cdot \frac{d}{dx}(g(x)) \blacktriangleright \frac{9\cdot (x+a)^2}{x^4}$

Question: 4.

Consider the functions $f(x)=-(1)^n(x+n)^n$ and $g(x)=\frac{1}{x^n}$, use CAS to verify $\left(f(x)g(x)\right)'=f'(x)g'(x)$

For the cases when $n=4$ and $n=5$.

Answer

$\text{Define } f(x)=(x+4)^4 \blacktriangleright \textit{Done}$

$\text{Define } g(x)=\frac{1}{x^4} \blacktriangleright \textit{Done}$

$\frac{d}{dx}(f(x)\cdot g(x)) \blacktriangleright \frac{-16\cdot (x+4)^3}{x^5}$

$\frac{d}{dx}(f(x))\cdot \frac{d}{dx}(g(x)) \blacktriangleright \frac{-16\cdot (x+4)^3}{x^5}$

$\text{Define } f(x)=-(x+5)^5 \blacktriangleright \textit{Done}$

$\text{Define } g(x)=\frac{1}{x^5} \blacktriangleright \textit{Done}$

$\frac{d}{dx}(f(x)\cdot g(x)) \blacktriangleright \frac{25\cdot (x+5)^4}{x^6}$

$\frac{d}{dx}(f(x))\cdot \frac{d}{dx}(g(x)) \blacktriangleright \frac{25\cdot (x+5)^4}{x^6}$

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Question: 5.

Consider the functions $f(x)=(x+10)^{10}$ and $g(x)=\dfrac{1}{x^{10}}$, use CAS to verify $\left(f(x)g(x)\right)'=f'(x)g'(x)$

Can you predict the general result? Hint use a slider for *n*..

Answer

$\text{Define } \mathbf{f}(x)=(-1)^{\mathbf{n}}\cdot(x+\mathbf{n})^{\mathbf{n}} \blacktriangleright \textit{Done}$

$\text{Define } \mathbf{g}(x)=\dfrac{1}{x^{\mathbf{n}}} \blacktriangleright \textit{Done}$ < > n =10.

$$\frac{d}{dx}\left(\mathbf{f}(x)\cdot \mathbf{g}(x)\right) \blacktriangleright \frac{-100\cdot(x+10)^9}{x^{11}}$$

$$\frac{d}{dx}\left(\mathbf{f}(x)\right)\cdot \frac{d}{dx}\left(\mathbf{g}(x)\right) \blacktriangleright \frac{-100\cdot(x+10)^9}{x^{11}}$$

$f(x)=-(1)^n(x+n)^n$ and $g(x)=\dfrac{1}{x^n}$, $\left(f(x)g(x)\right)'=f'(x)g'(x)=\dfrac{(-1)^{n-1}n^2(x+n)^{n-1}}{x^{n+1}}$

PART 2

Question: 6.

Consider the functions $f(x)=\dfrac{1}{1-x}$ where $x\in R\setminus\{1\}$ and $g(x)=x$, show that $\left(f(x)g(x)\right)'=f'(x)g'(x)$

Answer

$$f(x)=\frac{1}{1-x}=(1-x)^{-1},\quad g(x)=x$$

$$f'(x)=(1-x)^{-2}=\frac{1}{(1-x)^2}\quad g'(x)=1$$

$$\frac{d}{dx}\left(f(x)g(x)\right)=g(x)\frac{d}{dx}\left(f(x)\right)+f(x)\frac{d}{dx}\left(g(x)\right)$$

$$=\frac{x}{(1-x)^2}+\frac{1}{1-x}=\frac{x+(1-x)}{(1-x)^2}=\frac{1}{(1-x)^2}$$

$$f'(x)g'(x)=\frac{1}{(1-x)^2}$$

$\text{Define } \mathbf{f}(x)=\dfrac{1}{1-x} \blacktriangleright \textit{Done}$

$\text{Define } \mathbf{g}(x)=x \blacktriangleright \textit{Done}$

$$\frac{d}{dx}\left(\mathbf{f}(x)\cdot \mathbf{g}(x)\right) \blacktriangleright \frac{1}{(x-1)^2} ⚠$$

$$\frac{d}{dx}\left(\mathbf{f}(x)\right)\cdot \frac{d}{dx}\left(\mathbf{g}(x)\right) \blacktriangleright \frac{1}{(x-1)^2}$$

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Question: 7.

Consider $f(x)=\dfrac{1}{(2-x)^2}$ where $x\in R\setminus\{2\}$ and $g(x)=x^2$, show that $\left(f(x)g(x)\right)'=f'(x)g'(x)$

Answer

$$f(x)=\frac{1}{(2-x)^2}=(2-x)^{-2},\quad g(x)=x^2$$

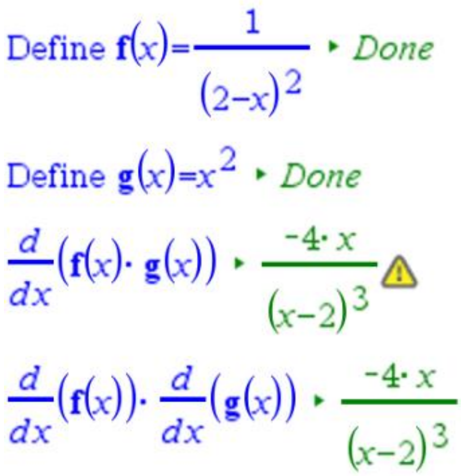
$$f'(x)=2(2-x)^{-3}=\frac{2}{(2-x)^3}\quad g'(x)=2x$$

$$\frac{d}{dx}\left(f(x)g(x)\right)=g(x)\frac{d}{dx}\left(f(x)\right)+f(x)\frac{d}{dx}\left(g(x)\right)$$

$$=\frac{2x^2}{(2-x)^3}+\frac{2x}{(2-x)^2}$$

$$=\frac{2x^2+2x(2-x)}{(2-x)^3}=\frac{4x}{(2-x)^3}$$

$$f'(x)g'(x)=\frac{4x}{(2-x)^3}$$



Question: 8.

Consider $f(x)=\dfrac{1}{(3-x)^3}$ where $x\in R\setminus\{3\}$ and $g(x)=x^3$, show that $\left(f(x)g(x)\right)'=f'(x)g'(x)$

Answer

$$f(x)=\frac{1}{(3-x)^3}=(3-x)^{-3},\quad g(x)=x^3$$

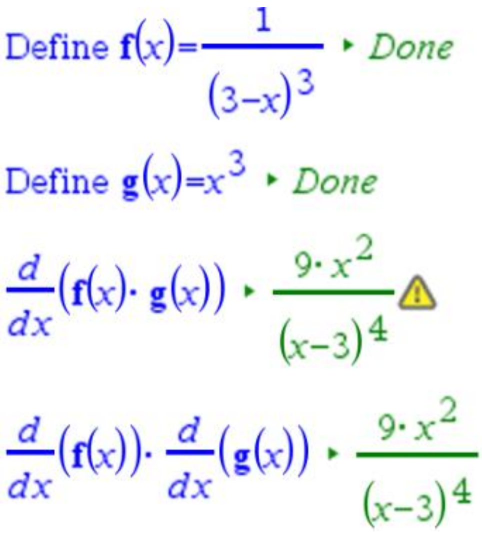
$$f'(x)=3(3-x)^{-4}=\frac{3}{(3-x)^4}\quad g'(x)=3x^2$$

$$\frac{d}{dx}\left(f(x)g(x)\right)=g(x)\frac{d}{dx}\left(f(x)\right)+f(x)\frac{d}{dx}\left(g(x)\right)$$

$$=\frac{3x^3}{(3-x)^4}+\frac{3x^2}{(3-x)^3}$$

$$=\frac{3x^3+3x^2(3-x)}{(3-x)^4}=\frac{9x^2}{(3-x)^4}$$

$$f'(x)g'(x)=\frac{9x^2}{(3-x)^4}$$



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Question: 9.

Consider $f(x)=\dfrac{1}{(b-x)^m}$ where $x\in R\setminus\{b\}$ and $g(x)=x^n$, given that $\left(f(x)g(x)\right)'=f'(x)g'(x)$

Express both *b* and *m* in terms of *n*.

Hence write generalised sets of functions $f(x)$ and $g(x)$ which satisfy $\left(f(x)g(x)\right)'=f'(x)g'(x)$,

If $f(x)=\dfrac{1}{(10-x)^{10}}$ where $x\in R\setminus\{10\}$ and $g(x)=x^{10}$ can you predict $\left(f(x)g(x)\right)'=f'(x)g'(x)$.

Using your conjecture is it true for non-integer values of *n*? Prove your conjecture in general.

Answer

$$f(x)=\frac{1}{(b-x)^m}=(b-x)^{-m},\quad g(x)=x^n$$

$$f'(x)=m(b-x)^{-m-1}=\frac{m}{(b-x)^{m+1}}\quad g'(x)=nx^{n-1}$$

$$\frac{d}{dx}(f(x))g(x)=g(x)\frac{d}{dx}(f(x))+f(x)\frac{d}{dx}(g(x))$$

$$\frac{d}{dx}(f(x))g(x)=\frac{mx^n}{(b-x)^{m+1}}+\frac{nx^{n-1}}{(b-x)^m}=\frac{mx^n+nx^{n-1}(b-x)}{(b-x)^{m+1}}=\frac{x^n(m-n)+nbx^{n-1}}{(b-x)^{m+1}}$$

$$f'(x)g'(x)=\frac{nmx^{n-1}}{(b-x)^{m+1}}$$

So that $b=m$ and $b=m=n$

In general $f(x)=\dfrac{1}{(n-x)^n}=(n-x)^{-n},\quad g(x)=x^n$ and $\left(f(x)g(x)\right)'=f'(x)g'(x)=\dfrac{(-1)^{n-1}n^2x^{n-1}}{(x-n)^{n+1}}$

Define $\mathbf{f}(x)=\dfrac{1}{(\mathbf{n}-x)^{\mathbf{n}}}$ ▸ *Done* $\mathbf{n}=10.$

Define $\mathbf{g}(x)=x^{\mathbf{n}}$ ▸ *Done*

$$\frac{d}{dx}(\mathbf{f}(x)\cdot\mathbf{g}(x)) \blacktriangleright \frac{-100\cdot x^9}{(x-10)^{11}}$$

$$\frac{d}{dx}(\mathbf{f}(x))\cdot\frac{d}{dx}(\mathbf{g}(x)) \blacktriangleright \frac{-100\cdot x^9}{(x-10)^{11}}$$

Note that negative values of *n* just produce the questions in Part A.

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Also this is satisfied for non-integer values of *n*.

n	$f(x)$	$g(x)$	$\left(f(x)g(x)\right)' = f'(x)g'(x)$
4	$\frac{1}{(4-x)^4}$	x^4	$\frac{-16x^3}{(x-4)^5}$
5	$\frac{1}{(5-x)^5}$	x^5	$\frac{25x^4}{(x-5)^6}$
10	$\frac{1}{(10-x)^{10}}$	x^{10}	$\frac{-100x^9}{(x-10)^{11}}$
$\frac{1}{2}$	$\frac{\sqrt{2}}{\sqrt{1-2x}}$	$\sqrt{x}$	$\frac{\sqrt{2}}{2\sqrt{x}(1-2x)^{\frac{3}{2}}}$
$-\frac{1}{2}$	$\frac{\sqrt{-2(2x+1)}}{2}$	$\frac{1}{\sqrt{x}}$	$\frac{\sqrt{-2(2x+1)}}{4x^{\frac{3}{2}}}$
$\frac{3}{2}$	$\frac{-2\sqrt{2}}{(3-2x)^{\frac{3}{2}}}$	$\sqrt{x^3}$	$\frac{9\sqrt{2x}}{(3-2x)^{\frac{5}{2}}}$

$$\text{Define } \mathbf{f}(x)=\frac{1}{(\mathbf{n}-x)^{\mathbf{n}}} \blacktriangleright \textit{Done} \qquad \mathbf{n}:=\frac{1}{2} \blacktriangleright \frac{1}{2}$$

$$\mathbf{f}(x) \blacktriangleright \frac{\sqrt{2}}{\sqrt{1-2\cdot x}} \qquad \mathbf{g}(x) \blacktriangleright \sqrt{x}$$

$$\text{Define } \mathbf{g}(x)=x^{\mathbf{n}} \blacktriangleright \textit{Done}$$

$$\frac{d}{dx}\left(\mathbf{f}(x)\cdot \mathbf{g}(x)\right) \blacktriangleright \frac{\sqrt{2}}{2\cdot \sqrt{x}\cdot (1-2\cdot x)^{\frac{3}{2}}}$$

$$\mathbf{n}:=\frac{3}{2} \blacktriangleright \frac{3}{2} \qquad \mathbf{f}(x) \blacktriangleright \frac{-2\cdot \sqrt{2}}{(2\cdot x-3)\cdot \sqrt{3-2\cdot x}}$$

$$\text{Define } \mathbf{g}(x)=x^{\mathbf{n}} \blacktriangleright \textit{Done} \qquad \mathbf{g}(x) \blacktriangleright x^{\frac{3}{2}}$$

$$\frac{d}{dx}\left(\mathbf{f}(x)\cdot \mathbf{g}(x)\right) \blacktriangleright \frac{9\cdot \sqrt{2\cdot x}}{(3-2\cdot x)^{\frac{5}{2}}}$$

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$$f(x)=\frac{1}{(n-x)^n}=(n-x)^{-n},\quad g(x)=x^n$$

$$f'(x)=n(n-x)^{-n-1}=\frac{n}{(n-x)^{n+1}}\quad g'(x)=nx^{n-1}$$

$$\frac{d}{dx}\left(f(x)g(x)\right)=g(x)\frac{d}{dx}\left(f(x)\right)+f(x)\frac{d}{dx}\left(g(x)\right)$$

$$=\frac{nx^n}{(n-x)^{n+1}}+\frac{nx^{n-1}}{(n-x)^n}$$

$$=\frac{nx^n+nx^{n-1}(n-x)}{(n-x)^{n+1}}$$

$$=\frac{n^2x^{n-1}}{(n-x)^{n+1}}$$

$$f'(x)g'(x)=\frac{n^2x^{n-1}}{(n-x)^{n+1}}$$

PART 3

Question: 10.

Consider the two non-constants functions $f(x)$ and $g(x)$, where $g(x)\neq g'(x)$ if $\left(f(x)g(x)\right)'=f'(x)g'(x)$ then show that $\frac{f'(x)}{f(x)}=\frac{g'(x)}{g'(x)-g(x)}$

Answer

$$\left(f(x)g(x)\right)'=\frac{d}{dx}\left(f(x)g(x)\right)$$

$$=f'(x)g(x)+f(x)g'(x)=f'(x)g'(x)$$

$$f(x)g'(x)=f'(x)g'(x)-f'(x)g(x)$$

$$f(x)g'(x)=f'(x)\left(g'(x)-g(x)\right)$$

$$\frac{f'(x)}{f(x)}=\frac{g'(x)}{g'(x)-g(x)}$$

Question: 11.

Consider the case when $g(x)=e^{kx},\ k\in R\setminus\{1\}$, solve the differential equation in Question 10 and hence find a function $f(x)$ which satisfies $\left(f(x)g(x)\right)'=f'(x)g'(x)$.

Answer

$$\int\frac{f'(x)}{f(x)}dx=\int\frac{g'(x)}{g'(x)-g(x)}dx\quad g(x)=e^{kx}$$

$$\log_e\left(f(x)\right)=\int\frac{ke^{kx}}{ke^{kx}-e^{kx}}dx=\int\left(\frac{k}{k-1}\right)dx=\frac{kx}{k-1}+c,\quad k\neq 1$$

$$f(x)=e^{\frac{kx}{k-1}}$$

$\text{Define } \mathbf{g}(x)=\mathbf{e}^{k\cdot x} \blacktriangleright \textit{Done}$

$\text{Define } \mathbf{dg}(x)=\frac{d}{dx}(\mathbf{g}(x)) \blacktriangleright \textit{Done}$

$\text{deSolve}\left(y'=\frac{\mathbf{dg}(x)}{\mathbf{dg}(x)-\mathbf{g}(x)} \text{ and } y(0)=0,x,y\right)$

$\blacktriangleright y=\frac{k\cdot x}{k-1}$

$\text{Define } \mathbf{f}(x)=\mathbf{e}^{\frac{k\cdot x}{k-1}} \blacktriangleright \textit{Done}$

$\frac{d}{dx}(\mathbf{f}(x)\cdot \mathbf{g}(x))$

$\blacktriangleright \left(\frac{1}{k-1}+k+1\right)\cdot \mathbf{e}^{\left(\frac{1}{k-1}+k+1\right)\cdot x}$ ⚠

$\text{propFrac}\left(\frac{d}{dx}(\mathbf{f}(x))\cdot \frac{d}{dx}(\mathbf{g}(x))\right)$

$\blacktriangleright \left(\frac{1}{k-1}+k+1\right)\cdot \mathbf{e}^{\left(\frac{1}{k-1}+k+1\right)\cdot x}$ ⚠

Question: 12.

Can you find other sets of functions for example non-polynomial functions $f(x)$ and $g(x)$ for example trigonometric exponential or logarithmic functions $f(x)$ and $g(x)$ which satisfy $\left(f(x)g(x)\right)'=f'(x)g'(x)$

Generalize your results.

Answer

$\text{Define } \mathbf{f}(x)=\mathbf{e}^{k\cdot x} \blacktriangleright \textit{Done}$

$\text{Define } \mathbf{g}(x)=a\cdot \sin(\mathbf{n}\cdot x)+b\cdot \cos(\mathbf{n}\cdot x) \blacktriangleright \textit{Done}$

$\frac{d}{dx}(\mathbf{f}(x)\cdot \mathbf{g}(x)) \blacktriangleright ((b\cdot k-3\cdot a)\cdot \cos(3\cdot x)+(-a\cdot k-3\cdot b)\cdot \sin(3\cdot x))\cdot \mathbf{e}^{k\cdot x}$

$\frac{d}{dx}(\mathbf{f}(x))\cdot \frac{d}{dx}(\mathbf{g}(x))$

$\blacktriangleright -3\cdot k\cdot (a\cdot \cos(3\cdot x)+b\cdot \sin(3\cdot x))\cdot \mathbf{e}^{k\cdot x}$

$\text{solve}(b\cdot k-3\cdot a=-2\cdot k\cdot a$ $\blacktriangleright a=0 \text{ and } b=0$

It is not possible to find trigonometric functions which satisfy the required results.

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