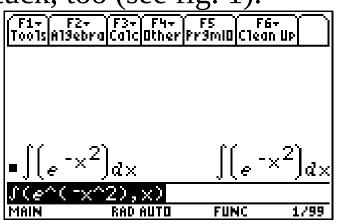
Solving Differential Equations via
The Fundamental Theorem of Calculus

By Dave Slomer

Some differential equations are downright easy to solve, such as $\dfrac{dy}{dx} = 3x^2$, which has the general solution $y = x^3 + C$. Many others are not nearly so easy and some are impossible to solve symbolically.

For example, suppose you have to solve the differential equation $\dfrac{dy}{dx} = e^{-x^2}$ with initial condition $y(1) = 2$. But you can't find an antiderivative of e^{-x^2}. You try your TI-89, but it's stuck, too (see fig. 1).

Fig. 1 

You can use numeric or graphical methods, such as ***Euler's method*** and ***slope fields***, to get a <u>feel</u> for the <u>nature</u> of the solution. These are built into your TI-89 in "differential equation graphing mode". Their visual clues about the solution of the given differential equation (defined in figure 2b) are shown in figure 2a. The dark function in figure 2a is the Euler's method solution to $\dfrac{dy}{dx} = e^{-x^2}$ if $y(1) = 2$. Call the solution $y = E(x)$. Since the graph of E is traceable, you could find out as much as you might want to about the solution, making a table of values by tracing to as many points as you care to.

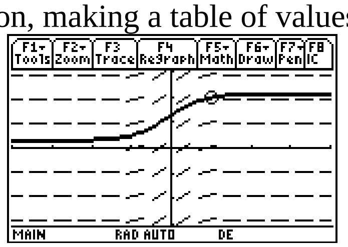
Fig. 2a 

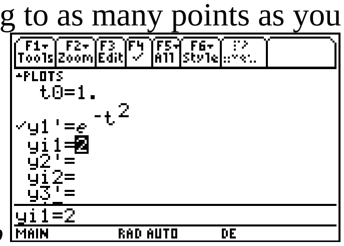
Fig. 2b 

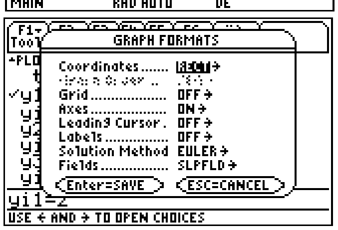
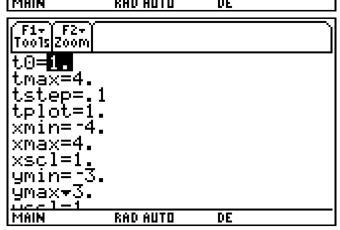
Fig. 2c 

Fig. 2d 

Exercise 1: Fill in the second column of Table 1 (below) with trace values from the Euler solution (y = E(x)). [Get into differential equation graphing mode (press [MODE] [▸] [6]) and set it up as shown above in figures 2b through 2d. To do so, press [Y=] ([◆] [F1]) and, make **y1'=e^(-x^2))** and set **t0=1** and **yi1=2** to establish the given initial condition $y(1)=2$. Then, press [◆] [I] and use the cursor to change the **GRAPH FORMATS** screen to match figure 2c. Finally, press [◆] [F2] and make the window match what is shown in figure 2d. Finally, press [◆] [F3] to see the graph and [F3] to trace]

Table 1:

x	***E(x) (approx. value)***		
-2			
-1			
0			
1			
2			

Exercise 2: Change the **Solution Method** from **EULER** to **RK** (refer to figure 2c). This method, short for "Runge-Kutta", offers better accuracy but is slower because of the added complexity of the method. Label the third column of the table above "RK(x)" and fill it in by tracing.

<u>*Yes, you DO have better things to do than fill in tables! Let the '89 do the work. But how?*</u>

While no "closed form" solution exists for antiderivatives of e^{-x^2}, the Fundamental Theorem of Calculus (which demonstrates that integration and differentiation are inverse operations) says that we can <u>always</u> find an antiderivative of <u>any</u> continuous function. So, $\int e^{-x^2}dx = \int_a^x e^{-t^2}\,dt$ for any choice of the constant a, since $y = e^{-x^2}$ is continuous for all x.

At first, this looks and feels like the ol' runaround. But, trusting your '89 to compute derivatives correctly, just key in **¶((e^(-t^2),t,a,x),x)** and read the result. The integral $\int_a^x e^{-t^2}\,dt$ is a function of x because x is the upper limit, so it makes sense to graph functions defined by equations such as $A(x) = \int_a^x e^{-t^2}\,dt$. Such functions are called *area functions*. So it makes sense to find the derivative of an area function with respect to x.

Exercise 3: Explain why result of **¶((e^(-t^2),t,a,x),x)** proves that $\int_a^x e^{-t^2}\,dt$ is an antiderivative of e^{-x^2}.

Of course, there are infinitely many antiderivatives of e^{-x^2}, as the slope field suggests (fig. 2a). Would the general antiderivative here be $\int_a^x e^{-t^2}\,dt$ because **a** is an unknown constant? Or should you add on the usual constant ***C***? This is answered in Exercise 10.

Meanwhile, for a specific value of the lower limit *a*, you might go to **Function** graphing $\boxed{\text{MODE}}$, store $\int_a^x e^{-t^2}\,dt$ into **y1**, and have the ’89 draw the graph. You could then also look at a table of values. [You could even try to find (or write) a TI-89 program that computes approximate values of definite integrals and run it for a bunch of *x*’s.]

But <u>how do you choose **a**</u>? (Forget about whether you need to add *C* for now. One thing at a time.) Determining **a** by trial-and-error is tempting (though not for long).

Define **y1⇒ (e^(-t^2),t,a,x)** and experiment with **a** on the home screen (as in figure 3a). This is easy enough, and who knows what it might tell us? Store 0 into **a** and recall **y1(x)** to see that **y1** looks right, as it does in figure 3a below.

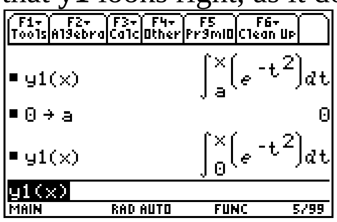
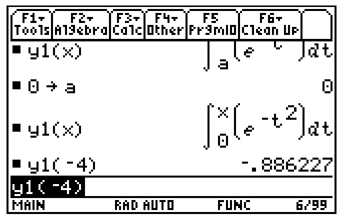
Fig. 3a 

Fig. 3b 

But if you graph, even if you set **xres=9** (meaning to plot points at every 9th pixel), you will wonder when that first point is going to show up. [If your ’89 has begun graphing, you can press $\boxed{\text{ON}}$ to stop it. Or let it crank. It’ll stop sooner or later, one way or another!] Computing **y1(-4)** [for example] on the home screen takes a fairly long time (explaining why graphing takes so long) but at least you get to see a value (fig. 3b).

So, try the ’89’s table feature, which only calculates 5 function values at a time. Plotting the 5 points will likely tell you whether **y1** is defined correctly. Make the table start at –2 and up by 1’s. [To do so press ◆ $\boxed{\text{F4}}$ and make **tblStart** = -2 and **Δtbl** = 1; then press ◆ $\boxed{\text{F5}}$ to see the table.] The table looks like so:

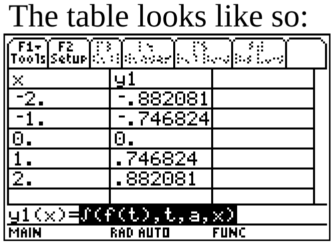
Fig. 4 

Figure 4 shows that **y1** does not pass through the initial point (1,2), so **a** = 0 is wrong, having caused **y1** to pass through (1,0.746824) [approximately], missing (1,2) by about 1.253 units. Adding that nasty approximation to **y1** is tempting (Could this be *C*?). But noting that (0,0) is on this area function and that those appear to be exact coordinates gives hope for an <u>exact</u> hit. It’s time to quit trial-and-error and analyze.

Exercise 4: You need **y1(1) = 2**. <u>Exactly</u> 2. How might you choose **a** in light of this?

Hint: Fill in the blank in the definition of **y1** *below with a number that will make* **y1(1)** *<u>exactly</u> equal 0, making (1,0) an <u>exact</u> point on the graph.*

y1⇒ (e^(-t^2),t,______,x)

Keep analyzing. Just make sure that **y1(1) = 2** (not 0) and that the derivative of **y1** with respect to x is e^{-x^2}. It won't take too long to find this unique specific solution correctly.

Exercise 5: Once you have found the right **a**, label column 4 of Table 1 "**y1(x)**" and fill it in. Compare it with the Euler's method and "RK" method values. You can do this on the home screen, one x at a time, or use the '89's table feature, as was done in figure 4.

Exercise 6: Analyze all four columns of Table 1. Make some guesses about the relative accuracy of Euler's method, the "RK" method, and the area function (**y1**) method. Draw some well-thought-out conclusions.

Exercise 7: The differential equation $\frac{dy}{dx} = \ln(\sin(x))$ (subject to the condition that $x = 2$ when $y = 3$) has exact solution $y =$ _____ + $\int_{\_}^{\_}$ ______ dt.

(Check: Does ($y(2) = 3$? Does $\frac{dy}{dx} = \ln(\sin(x))$? [Feel free to let your '89 compute the derivative.])

Exercise 8: Find the antiderivative of $y = \sin(x^2)$ that passes through the point (4,2) and graph it. Is the derivative of your solution equal to $\sin(x^2)$?

Exercise 9: Find the derivative with respect to x of $\int_2^x g(t)dt$ and $\int_x^2 g(t)dt$ (assume that g is suitably continuous).

Exercise 10: In **<u>general</u>**, $\frac{dy}{dx} = f(x)$ (subject to the conditions that $y = b$ when $x = a$ and that f is properly continuous) has solution $y =$ _____ + $\int_{\_}^{\_}$ ______ dt, whether or not $f(x)$ has a "closed form" antiderivative.

Calculus Generic Scope and Sequence Topics: Differential Equations, Antiderivatives
NCTM Standards: Number and operations, Algebra, Geometry, Measurement, Problem solving, Connections, Communication, Representation