NUMB3RS Activity: The Orchard Problem
Episode: “Brutus”

Topic: Lattice points in the coordinate plane **Grade Level** 9 - 11
Objective: Find relationships among lattice points in a grid
Time: 30 minutes
Materials: TI-83 Plus/TI-84 Plus graphing calculator (optional)

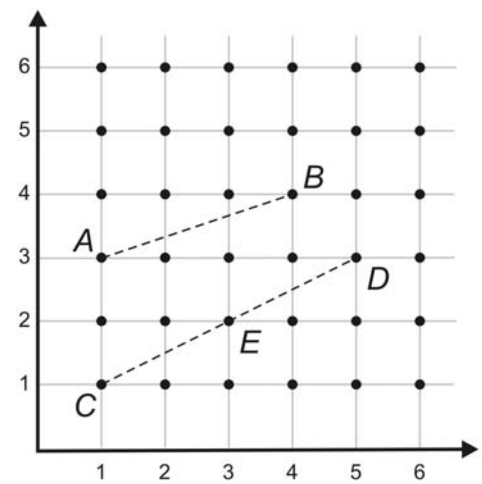
Introduction

When Charlie’s facial recognition program fails to find any matches when searching for a renegade CIA agent, he remembers Euclid’s Orchard problem. “I kept coming up with zero matches. Then I remembered the classic example of Euclid’s Orchard. When we look at an orchard from the sky, we can see all of the trees that comprise it. If we stand on the ground, however, our view is considerably different. Some of the trees block your view of others. This means we were looking for someone who had access to all of the CIA program files—instead we should be looking at people who only had access to some of them. At first, this seems like it would increase our suspect list—but, in fact, it dramatically decreases it.” This activity explores ideas related to the Orchard problem.

Discuss with Students

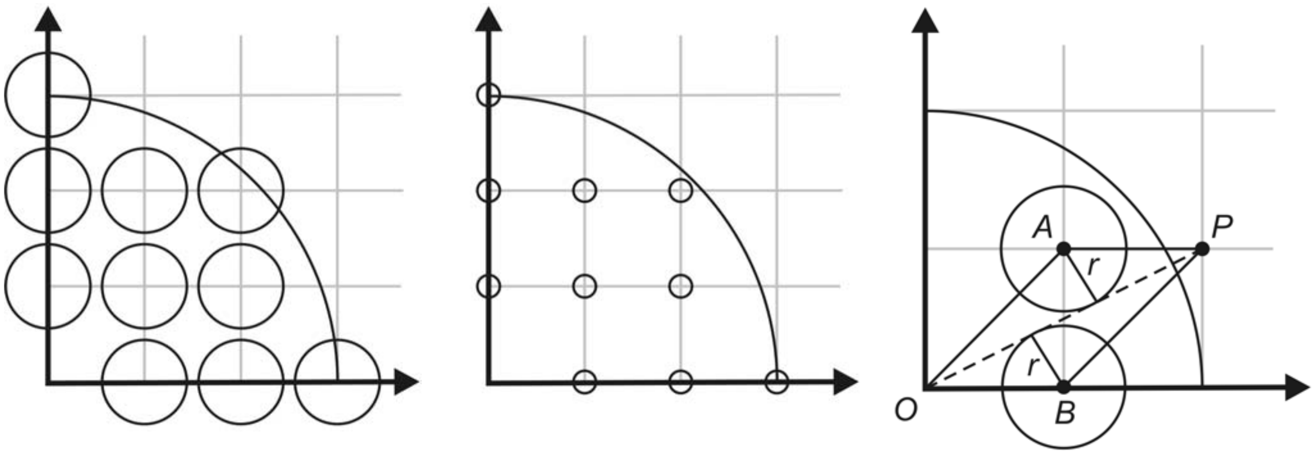
The Orchard problem imagines planting a “tree” at each lattice point in a (coordinate) orchard. A **lattice point** is one whose coordinates are integers.

In the first part of the activity, imagine the trees as lattice points in a square orchard and an observer located at another lattice point. Lines of sight are segments joining lattice points. Two lattice points $A(a, b)$ and $B(c, d)$ are **mutually visible** if the line segment joining them contains no further lattice points. An observer at *A* can see a tree at *B,* but an observer at *C* cannot see a tree at *D* because there is a tree at *E* blocking his view. Determining whether two points are mutually visible requires using slope to determine whether there is another lattice point on the segment joining the two points. Another setting for the problem is to imagine taking pictures of a marching band from various locations so that the set of pictures includes all members of the band. Ideally the entire band could be photographed from one location.



It is possible to use the TI-83 Plus/TI-84 Plus calculator to represent trees as grid points (set an appropriate window, and turn the grid “on”) and then use the Line option in the Draw menu to draw segments between points.

In the second part of the activity, an introduction to a version of Euclid's Orchard problem, the trees (identical vertical cylinders with the same radius) are planted at each lattice point inside a circular orchard with center (0, 0) where the observer is located. Lines of sight are rays containing the origin. Given the radius of the orchard, *R*, the problem is to determine the smallest value of the radius, *r*, of each tree so that the observer is unable to see out of the orchard in any direction. For simplicity, only the portion of the orchard in the first quadrant is considered.



In the two orchards with *R* = 3 shown above, an observer is clearly unable to see out of the orchard on the left but can easily see out of the orchard on the right. Assuming that the observer cannot see out if the line of sight is tangent to a tree, then the third figure shows *r* when *R* = 2.

Student Page Answers:
1. *13 points: (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 2), (4, 4), (4, 6), (5, 5), (6, 2), (6, 3), (6, 4), (6, 6).*
2. *When the slope b/a of the line joining (0, 0) and (a, b) is not in simplest form or equivalently when the greatest common divisor of a and b, GCD (a, b, ≠ 1.* **3.** *13 points: (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (4, 2), (4, 4), (4, 6), (5, 3), (5, 6), (6, 2), (6, 4), (6, 6)* **4.** *GCD(a – 2, b) ≠ 1* **5a.** *All segments containing (6,4) and each of the 9 points in the grid contain no other lattice points.* **5b.** *(4, 6) is one answer.* **6.** *The area of parallelogram OAPB is 1 which equals twice the area of △OAP. However, the area of △OAP = (1/2)(r)* $\sqrt{5}$ *so that r = 1/*$\sqrt{5}$ **7.** *Choose P to be the point with y-coordinate 1 that is nearest the circle of radius R. Then r = 1/*$\sqrt{10}$ *; r = 1/*$\sqrt{17}$ ***8.*** *r = 1/*$\sqrt{R^2+1}$ *.*

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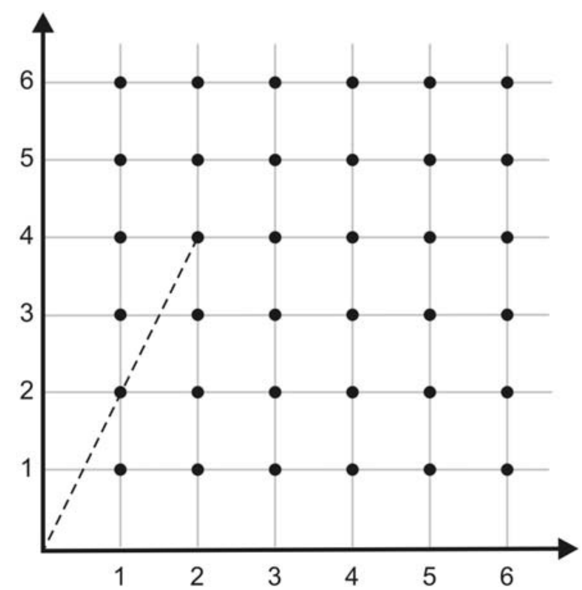
When Charlie's facial recognition program fails to find any matches when searching for a renegade CIA agent, he remembers Euclid's Orchard problem. "I kept coming up with zero matches. Then I remembered the classic example of Euclid's Orchard. When we look at an orchard from the sky, we can see all of the trees that comprise it. If we stand on the ground, however, our view is considerably different. Some of the trees block your view of others." This activity explores ideas related to the Orchard problem.

Imagine a point represents a tree. Plant a tree at each of the 36 points with integer coordinates in the square orchard whose corners are (1, 1), (6, 1), (6, 6), and (1, 6). A point with integer coordinates is known as a **lattice point**. Now imagine you are an observer located at a lattice point outside this orchard. Which trees can you see? Assume that your line of sight is the segment joining you and a tree.



1. Suppose you are located at (0, 0). Which trees (lattice points) in this orchard with $1 \le x \le 6$ and $1 \le y \le 6$ are you **unable** to see? For example, you cannot see the tree at (2, 4) since it is blocked by the tree at (1, 2).

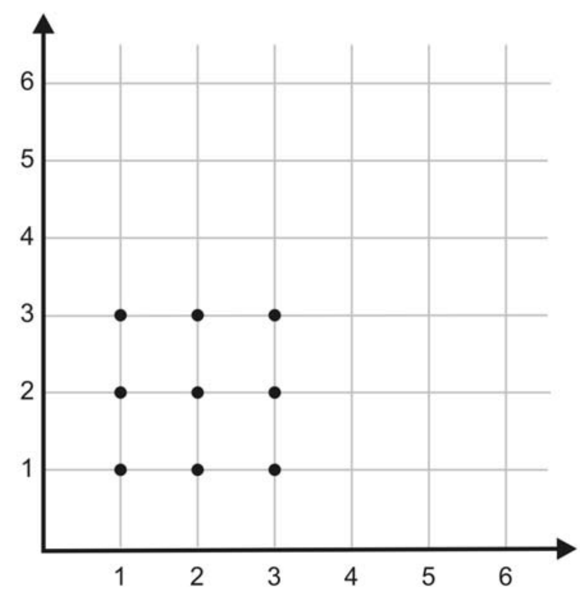
More formally, two lattice points (*a, b*) and (*c*, *d*) are **mutually visible** if the line segment joining them contains no further lattice points

2. Under what condition on *a* and *b* will the point (*a*, *b*) **not** be visible from the origin?

3. Which lattice points in this grid with $1 \le x \le 6$ and $1 \le y \le 6$ are **not** visible from the external point (2, 0)?

4. Under what condition on *a* and *b* will the point (*a*, *b*) not be visible from the point (2, 0)? Hint: One strategy is to examine the slope of the segment joining (2, 0) and (*a* ,*b*).

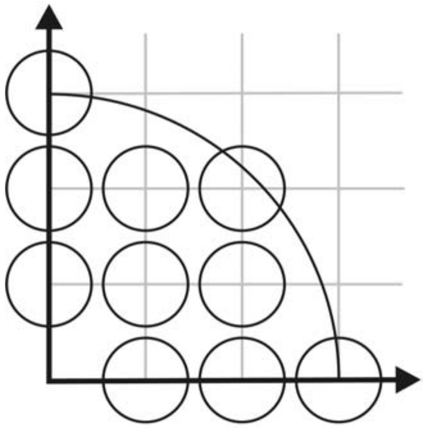
Just as Charlie restricted his search to agents who had access to only some of the files, restrict your search to a smaller orchard where the trees are planted at lattice points with $1 \le x \le 3$ and $1 \le y \le 3$. Is there a lattice point outside this smaller orchard where you could be located and see all of the trees?

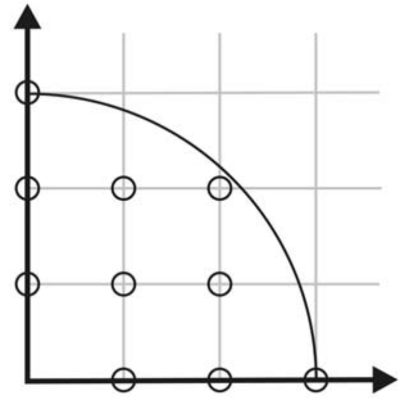


5. a. Explain why (6, 4) is such a point. That is, why are all lattice points in the grid with $1 \le x \le 3$ and $1 \le y \le 3$ visible from (6, 4)?

b. Find another point from which all lattice points in this grid are visible.

In Euclid's Orchard problem, the trees, which are represented as identical vertical cylinders with the same radius, are planted at each lattice point inside a circular orchard with center (0, 0) where the observer is located. Lines of sight are rays containing the origin.

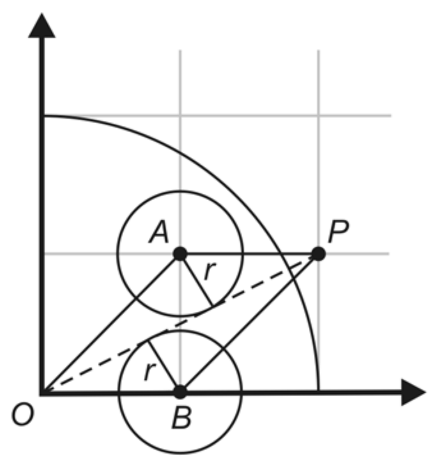




In the two orchards with radius $R = 3$ shown above, an observer standing at the origin is clearly unable to see out of the orchard on the left but can easily see out of the orchard on the right.

Given the radius of the orchard, R, the problem is to determine the smallest value of the radius, r, of each tree so that the observer is unable to see out of the orchard in any direction. For simplicity, only the portion of the orchard in the first quadrant is considered.

Assuming that the observer cannot see out if the line of sight is tangent to a tree, then this figure shows r when $R = 2$.



6. Find the smallest possible value of the radius r for these two trees in the orchard with $R = 2$. (**Hint:** Examine the area of *OAPB* in two ways. Why is it sufficient to consider only this line of sight?)

7. Find the analogous value of r for two such trees in orchards of radius 3 and 4. Explain your strategy. You may want to make a sketch.

8. Make a conjecture about the general relationship between the smallest possible value of r, the radius of the trees, and R, the radius of the orchard where the observer is unable to see out of it in any direction.

The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

1. Prove that all trees in a square orchard with corners (1, 1), (s, 1), (s, s), and (1, s) are visible by an observer located at (s^2, s + 1).

2. Examine the following Web sites and discover the connection between lattice points visible from the origin and Farey Sequences or Series:
http://mathworld.wolfram.com/FareySequence.html
http://www.cut-the-knot.org/ctk/PickToFarey.shtml

3. Look at "Euclid's Orchard" at the Web site below. Make sure you experiment with the applet and examine the orchard in all positions. George Polya and Gabor Szego first posed the problem in the 1920's.
http://mathworld.wolfram.com/EuclidsOrchard.html

Related Problem

Charlie made use of an analogy between Euclid's Orchard problem and the difficulty in his facial recognition program. As Polya writes in *Induction and Analogy in Mathematics*, "analogy is sort of a similarity. …consider the hand of a man, the paw of a cat, the foreleg of a horse, the fin of a whale, and so on." The problem solving strategy "Use a Simpler Analogous Problem" is a powerful one. One use of this strategy is to consider analogous relationships between two- and three-dimensional figures.

- How do you locate the center of a tetrahedron? What is an analogous problem in two dimensions?

- Formulate a theorem analogous to the Pythagorean theorem in three dimensions. One possibility to investigate is DeGua's theorem.

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