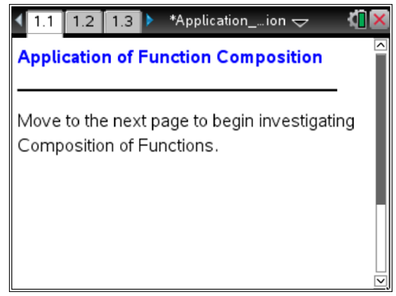
Application of Function Composition



Math Objectives

- Given functions f and g, the student will be able to determine the domain and range of each as well as the composite functions defined by $f(g(x))$ and $g(f(x))$.
- Students will interpret the graphical relationship between two functions and their composition.
- Students will determine relationships between two functions to create a composition of functions in the context of real-world applications.

Vocabulary

- domain
- range
- composite functions

About the Lesson

- This activity will explore the composition of functions, their graphs, and their application to real-world situations. As a result, students will:
  - Interpret relationships between domain and range of two functions that produce a composite function.
  - Analyze the data from a table that is generated from a model of a real-life situation that represents a composite function.
  - Summarize the relationship between functions and their composite function.

TI-Nspire™ Navigator™ System

- Use Class Capture and Live Presenter to examine examples of function composition.
- Use Quick Poll questions to adjust the pace of the lesson according to student understanding.

TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Graph a function.
- Play an animation.

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- Once a function has been graphed, the entry line can be shown by pressing ctrl G.

Lesson Files:
Student Activity

- Application_of_Function_ Composition.Student.pdf
- Application_of_Function_ Composition_Student.doc

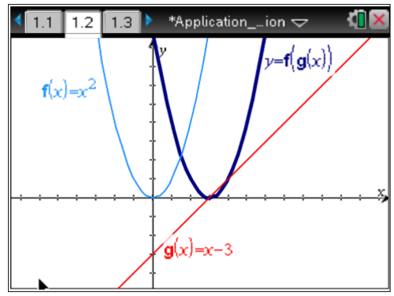
TI-Nspire document

- Application_of_Function_ Composition.tns
- Application_of_Function_ Composition_Teacher.tns

Visit www.mathnspired.com for lesson updates.

Discussion Points and Possible Answers

Teacher Tip: It is important to remind students here that $(f \circ g)(x) = f(g(x))$ and that $(g \circ f)(x) = g(f(x))$.



Move to page 1.2.

1. The composition of two functions is when you evaluate one function in terms of another function whereby the range of the first function determines the domain of the second function. This page shows the graphs of $f(x) = x^2$, $g(x) = x - 3$, $(f \circ g)(x) = f(g(x)) = (x-3)^2$. The function $(f \circ g)(x)$ is the composition of the functions *f*(*x*) and *g*(*x*). Describe the graph of the functions *f*(*x*), *g*(*x*) and $(f \circ g)(x)$ by answering the following questions.

   a. How does the composition function *f*(*g*(*x*)) affect the graph of *f*(*x*) and *g*(*x*)? Explain.

   Answer: Students should notice that the composition function *f*(*g*(*x*)) is shifted 3 units to the right. In terms of understanding what is happening within the actual composition, discuss with students how the quadratic function determines the shape and that the linear term creates a shift. This change in a function in terms of another is the crux of student understanding of this topic. It should not be overlooked and is quite important for student comprehension and analysis.

   b. Using the graphs generated, determine the domain and range of *f*(*x*), *g*(*x*), and *f*(*g*(*x*)).Explain your reasoning.

   Answer: The domain of f is all real numbers. The range of *f* is (0, ∞). The domain of *g* is all real numbers. The range of *g* is all real numbers. The domain of $(f \circ g)(x)$ is all real numbers. The range of $(f \circ g)(x)$ is (0, ∞). Students should explore the idea of input and output values for each function that includes the composite function and that its domain is equal to the range of *g*.

   c. From the graph, what are the domain and range of $(g \circ f)(x)$? How does this differ from your answer in part a? Explain your reasoning.

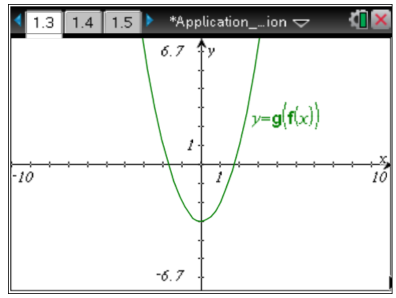
   Answer: The domain of $(g \circ f)(x)$ is all real numbers and the range is [0, ∞). Students should begin to explore the ideas of how the range of *f* affects the range of $(g \circ f)(x)$ and in general how the inside function affects the range of the outside function.

d. What type of graph will the function $(g \circ f)(x)$ produce? Explain your reasoning.

Answer: Students should recognize that the composition will yield a quadratic and that the composed graph will be translated down 3 units on the *y*-axis.

Move to page 1.3.

2. This page shows the graph of $(g \circ f)(x)$.



a. Explain how the composition between *f* and *g* translates the graphs of each function into the new function $(g \circ f)(x)$ and its graph.

Answer: Students should notice that the quadratic determines the shape and the graph is translated down 3 units. Students might ask "why" and point out how the graph of $g(f(x))$ differs from that in question 1a. This is a great time to discuss the order in which the functions are evaluated. Students should recognize that by evaluating *g* second, the constant term in $g(x) = x – 3$ shifts the graph down 3 units. If necessary, considerable time may be spent here discussing the evaluation of functions in terms of other functions.

b. Why is the graph not the same as $(f \circ g)(x)$? Explain your reasoning.

Answer: Students should realize that $(g \circ f)(x)$ and $(f \circ g)(x)$ are not commutative and therefore are not equal. In addition, it would be nice if students recognized that the process of transforming the function takes place based on the order in which the composition takes place.

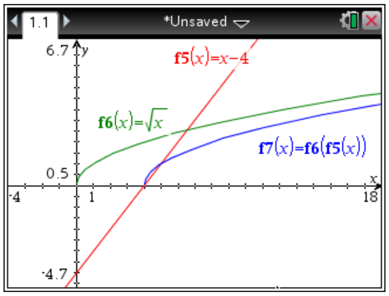
3. Now consider the functions $k(x) = x - 4$ and $m(x) = \sqrt{x}$. Find $m(k(x))$ algebraically. What is the domain and range of $m(k(x))$? Explain.

Answer: The domain for $m(k(x))$ is $[4, \infty)$ and the range is $[0, \infty)$. This is a good opportunity to discuss how the domain of $m(x)$ and $k(x)$ relate to one another. Notice that the range of $k(x)$ is all real numbers, but the domain of $m(x)$ is only defined where $x \geq 0$. So the possible values of the domain for $m(x)$ are all real numbers, but upon further consideration, $m(k(x))$ is only defined when $k(x) \geq 0$.

TI-Nspire Navigator Opportunity: *Class Capture* or *Live Presenter*
See Note 1 at the end of this lesson.

Move to page 1.4.

4. Graph functions $k(x) = x - 4$, $m(x) = \sqrt{x}$, and $m(k(x))$ on the graphs page by moving the cursor to the entry line at the bottom of the screen. You may graph these functions as *f5*(*x*), *f6*(*x*), and *f7*(*x*) respectively. Explain how the graphs of *m*(*x*), *k*(*x*), and *m*(*k*(*x*)) relate to the algebraic solution for finding the domain and range you found in question 3.



Answer: Students should recognize that the since *m*(*x*) is a square root function, it is only defined for values of *x* that make the radicand positive or 0. Therefore, whenever the radicand, *k*(*x*), is positive, the function *m*(*k*(*x*)) is defined.

> **Tech Tip:** It takes a few seconds to redraw some of the screens when using them on the handheld. The same actions can be done more quickly using the computer software. Give students time to explore the documents.

5. You have already discovered that the relationship between the composition of two functions is that the range of the inside function determines the domain of the outside function.

There are many real-life applications of composition of functions that occur. One such application is change in the area of a circular oil spill. Oil is spilling from a well at a rate of 2,500 gallons per day. At this rate the radius of the circular pool of oil will change at a constant rate of 4 miles per day.

a. Write an expression that represents the radius of the oil slick at time *t*, *r* days after the leak has started.

Answer: Students should recognize that the radius at time *t* is given by *r*(*t*) = *4t*

b. What are the domain and the range of the radius? Explain your reasoning.

Answer: Answers should reflect students' understanding of what is happening with the radius and that the domain must be [0, ∞) and the range [0, ∞). It is important to explain that the idea of domain and range must be taken in the context of the problem. In particular, this application has only positive real numbers for possible solutions since the distance of the radius cannot be negative.

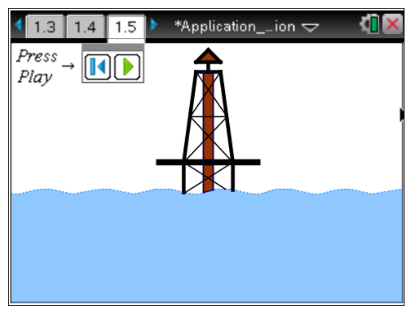
c. Write an expression to represent the area in terms of the radius and explain how the relationship between the radius and the area of the pool of oil represent the composition of the two functions.

Answer: Students should recognize that $A(r(t)) = 16\pi t^2$. Acceptable answers should include any correct variation of this formula, such as $A = \pi r^2$, $A(r) = \pi r^2$, or $A(t) = \pi t^2$. It is expected that students will begin to recognize that the relationship of the change in area is dependent upon the radius as a function of time.

d. What are the domain and range of the expression you created to represent the area of the oil spill? Explain your reasoning.

Answer: Students should recognize that the domain and range of the area of the spill have to be [0, ∞). Discussing the details of domain and range can be quite powerful for students. Stress the idea of the relative nature of the domain and range in terms of the reasonableness of the situation.

Move to page 1.5.



Tech Tip: Have students click anywhere on page 1.5 before selecting the **Play** button on the animation. This will collect the data needed for analysis of composition of functions.

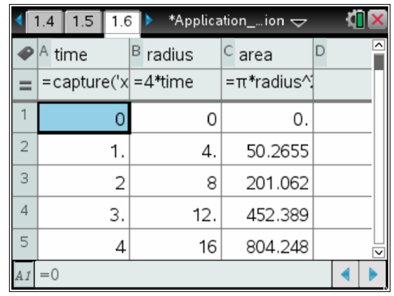
6. Page 1.5 contains an animation of an oil spill. By pressing play on the animation, data will be collected in a spreadsheet on page 1.6. Describe mathematically how the animation models the oil spill.

   Answer: A description of area changing at 4 miles per day and the area as a function of the radius and/or time is expected.

Tech Tip: On page 1.5, the animation will play over and over until "pause animation" is clicked.

Move to page 1.6.

7. Page 1.6 contains a spreadsheet that represents the situation of oil spilling from an oil well in a large body of water. In this case, the area of the circular spill is a composition of the functions of change in radius and elapsed time. The data you collected from step 6 is the data that should appear in the spreadsheet.



a. Explain how the data collected in the spreadsheet represents the functions you defined in questions 5a and 5b.

Answer: Students should recognize that "radius" = $4t$ and that "area" = πr^2.

b. What will be the shape of the graph of the function for $r(t)$? How do you know? Explain your reasoning.

Answer: Students should suggest that the radius is $r(t) = 4t$, which represents a straight line.

c. What will be the shape of the graph of the function for $A(r)$? How do you know? Explain your reasoning.

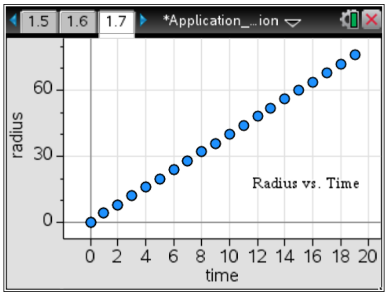
Answer: Students should present an answer of $A(r) = \pi r^2$. The reasoning should focus on the area of the circle being a quadratic expression.

TI-Nspire Navigator Opportunity: *Class Capture* or *Live Presenter*
See Note 2 at the end of this lesson.

Move to page 1.7.

8. Page 1.7 contains an empty graph. Move the cursor on the handheld to the bottom of the screen where it says "click to add the variable" and select the variable that represents the independent variable. Press tab and click to add the dependent variable on the vertical axis. This should represent the radius as a function of time. Did your conjecture from question 7b hold true for the shape of the graph? Explain why or why not.

Answer: Students should have the graph to the right. The graph should be linear. The conjecture from question 7 is an integral part of students' understanding of the student. Students should be on the right track with the understanding that the relationship between the radius and time is linear. Their reasoning should follow from the conjecture and be verified using the graph.

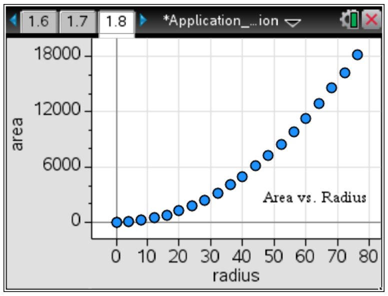


Teacher Tip: If students struggle to understand the relationship between time and radius, it might be helpful to give them a visual of what is happening. This might include pouring water out of a bottle at a constant rate. The size of the radius directly affects the time of pouring.

Move to page 1.8.

9. Page 1.8 contains an empty graph. Repeat the steps in question 8. Move the cursor on the handheld and click to add the variable that represents the independent and dependent variables for the area as a function of the radius. Did your conjecture from question 7c hold true for the shape of the graph? Explain why or why not.

Answer: Again, students should have developed an understanding that the area is a function of the radius. Students' graphs should match the graph to the right.

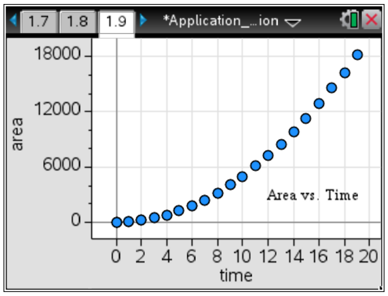


Move to page 1.9.

10. Part of the discovery of composition of functions involves realizing that the range of the inside function, in this example $r(t)$, becomes the domain for the outside function, $A(r(t))$. What do you think the graph of $A(t)$ should look like? Explain.

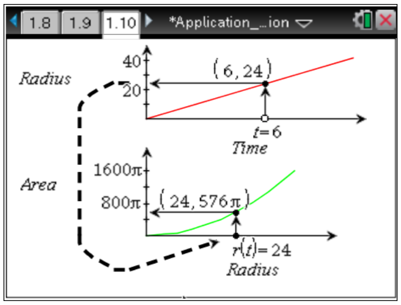
On this page, move the cursor on the handheld and click to add the variable that represents the independent variable, time, and dependent variable, area, for the area as a function of time. What do you notice about the shape of the graph when compared to the shape of the graph when area was a function of the radius? Explain.

Answer: Students should recognize that the area of the circle is a function of radius in terms of time. Therefore, area is a function of time. The conjecture should be that the graph will be quadratic and should match the graph to the right.



TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 3 at the end of this lesson.

Move to page 1.10.



11. To further explore the concept of how the area of the circular pool of oil is a function of the radius, which in turn is a function of time, and that the area is the composition of the functions of time and radius, use the handheld to move the open circle that represents time. Since you have discovered that $r(t) = 4t$ and that $A(r) = \pi r^2$, find a function that models the mathematical relationship of the area in terms of time.

Answer: Ideally, students would create a function of the form $(A \circ r)(t) = A(r(t)) = A(t)$ or $A(r(t)) = \pi(4t)^2$.

Teacher Tip: This page can be used to reinforce student understanding or to help students who are struggling. You might ask questions, such as "When time = 8, what is the area of the spill?" Encourage students to explore this page in detail.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- How two functions can be composed to create a new function.
- How the domain and range of the composition of two functions are interrelated.
- How to explain the relationship between the composition of two functions and their graphic representation.

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Note 1

Entire Document, *Class Capture* or *Live Presenter*: Pick a student at random to be Live Presenter to demonstrate how to enter the functions for number 4.

Note 2

Entire Document*, Class Capture* or *Live Presenter*: Use this opportunity to demonstrate how to graph all of the different relationships.

Note 3

Question 11, *Quick Poll*: Use several *Quick Polls* to verify that students understand the relationships demonstrated on page 1.10.

