NUMB3RS Activity: Pythagorean Triples
Episode: "Traffic"

Topic: The Pythagorean Theorem **Grade Level:** 9 - 12
Objective: Explore methods of generating Pythagorean Triples
Materials: a calculator
Time: 30 - 40 minutes

Introduction

In "Traffic," the FBI enlists Charlie's help after a series of seemingly random highway attacks grips Los Angeles. Larry is working at the chalkboard on some complicated looking mathematics when Charlie comes in and realizes that Larry is using the Pythagorean Theorem. This activity is designed for students with various levels of familiarity with the Pythagorean Theorem.

If students are familiar with the theorem, it is likely that they already know that a triangle with sides 3, 4, and 5 is a right triangle. Likewise, they probably know other sets of whole numbers that satisfy the Pythagorean Theorem (called *Pythagorean Triples*). How can Pythagorean Triples be found? In this activity, a common method is presented with the opportunity to explore properties of these triples. A less familiar method, using Fibonacci sequences, is also included.

Discuss with Students

The Pythagorean Theorem is fundamentally important to many mathematical topics that students are likely to study. It is not surprising that Larry uses it to explore more complicated mathematics and physics. Even in high school, it is used to find the lengths of sides of right triangles, to compute distance between points, to determine the equation of a circle, and to explore many topics in trigonometry.

If necessary, after Question #6 review the Fibonacci sequence with your students. The Fibonacci sequence starts with the whole numbers 0 and 1, then each successive term in the sequence is the sum of the previous two terms. Thus, the first nine terms of the sequence are 0, 1, 1, 2, 3, 5, 8, 13, and 21. Generally for any Fibonacci number F_n, $F_n = F_{n-1} + F_{n-2}$. Consistent with the activity, $F_{n+3} = F_{n+1} + F_{n+2}$.

Student Page Answers:
1. *Yes, one triangle with a third side of 6.* **2.** *One has a third side of length 5. The other has a third side of* $\sqrt{313} \approx 17.69$ **3.** *Answers will vary, but are likely to include {3, 4, 5}.* **4.** *Sample answers: {8, 15, 17}, {20, 21, 29}, {12, 16, 20}, {9, 40, 41}.* **5. and 6.** *Each of the Pythagorean Triples contains at least one number divisible by 2, 3, 4, and/or 5. Not all of them will have numbers divisible by 6 or 7.* **7.** *a = 272, b = 546, and c = 610 and they form a Pythagorean Triple.* **8.** *Answers will vary, but each pair will generate a Pythagorean Triple.* **Extension 1.** *Informally, if a is odd, then* a^2 *is odd. If b is odd, then* b^2 *is odd. If both a and b are odd, then* $a^2 + b^2$ *represents the sum of two odd numbers which is even. So* c^2 *must be even and therefore c must be even.* **Extension 2.** *The Web site* ***http://www.cut-the-knot.org/pythagoras/pythTripleDiv.shtml*** *contains a proof.* **Extension 3.** *Answers will vary. Examples include: {9, 12, 15}, {18, 24, 30} (or any other multiples of {3, 4, 5}), and {15, 31, 39}.*

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Name: ______________________________ Date: ______________

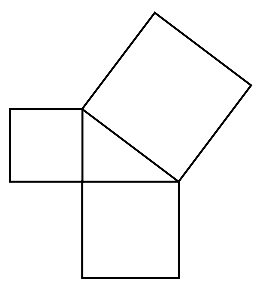
NUMB3RS Activity: Pythagorean Triples

In “Traffic,” the FBI enlists Charlie's help after a series of seemingly random highway attacks grips Los Angeles. Larry is working at the chalkboard on some complicated looking mathematics when Charlie comes in and realizes that Larry is using the Pythagorean Theorem.

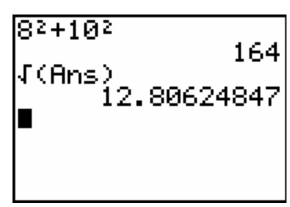
The Pythagorean Theorem is fundamentally important to a wide variety of mathematical applications. The theorem is named for Pythagoras of Samos (c. 560–480 B.C.) who was a Greek mathematician, natural philosopher, and religious leader. The theorem was known long before Pythagoras was even born – the Pythagorean Theorem was found on a Babylonian tablet that dates from 1600 B.C. or earlier.

The Pythagorean Theorem states that in a right triangle the sum of the squares of the lengths of two legs is equal to the square of the length of the hypotenuse (the hypotenuse of a right triangle is the side opposite the right angle). Algebraically, if a and b are the lengths of the legs and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.

Another way of thinking about the Pythagorean Theorem is to consider squares drawn using the sides of a right triangle as shown in the diagram below. The sum of areas of the two squares along the perpendicular sides is equal to the area of the square along the hypotenuse.



Consider a right triangle in which one leg is 8 and the other is 10. We can use the Pythagorean Theorem to find the length of the hypotenuse.



1. Are there other right triangles that have side lengths (not necessarily legs) of 8 and 10? If so, find the length of the third side for these triangles.

2. There is more than one right triangle that has sides of lengths 12 and 13. Use the Pythagorean Theorem to find the length of the third side for each of these triangles.

3. Any set of three whole numbers that satisfy the Pythagorean Theorem is called a *Pythagorean Triple*. Examples of Pythagorean Triples include {6, 8, 10}, {5, 12, 13}, and {7, 24, 25}. Find another Pythagorean Triple. In other words, find another set of positive integers a, b, and c such that $a^2 + b^2 = c^2$.

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Besides using a "guess and check" strategy, there is an algebraic way to create Pythagorean Triples. Take any two different whole numbers, m and n, such that $m > n$. First, find the values of $m^2 - n^2$, $2mn$, and $m^2 + n^2$. Call these three values a, b, and c (the side lengths of a right triangle). Notice that $a^2 + b^2 = c^2$. These three numbers form a Pythagorean Triple. To prove that this is always true, show that $\left(m^2 - n^2\right)^2 + \left(2mn\right)^2 = \left(m^2 + n^2\right)^2$ by expanding and simplifying both sides of the equation.

$$\left(m^2 - n^2\right)^2 + \left(2mn\right)^2 = \left(m^2 + n^2\right)^2$$
$$m^4 - 2m^2n^2 + n^4 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4$$
$$m^4 + 2m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4$$

Example: Suppose that $m = 10$ and $n = 6$. Then $m^2 - n^2 = 64$, $2mn = 120$, and $m^2 + n^2 = 136$. Use your calculator to check that these form a Pythagorean Triple.

4. Choose different values of m and n to fill in five different Pythagorean Triples in the table below.

m	***n***	***a***	***b***	***c***	**Div. by 2**	**Div. by 3**	**Div. by 4**	**Div. by 5**	**Div. by 6**	**Div. by 7**
10	6	64	120	136						

5. Check off the rest of the columns of the table to answer the following: how many of your Pythagorean Triples contain at least one number that is even? How many contain at least one number that is divisible by 3? by 4? by 5? by 6? by 7?

6. Compare your answers to Question #5 with a classmate. Describe any patterns you see.

Another method for generating Pythagorean Triples is based on the use of four consecutive Fibonacci numbers as follows: Let $a = F_n \cdot F_{n+3}$; $b = 2 \cdot F_{n+1} \cdot F_{n+2}$; and $c = F_{n+1}^{\,2} + F_{n+2}^{\,2}$, where F_n, F_{n+1}, F_{n+2}, and F_{n+3} are consecutive Fibonacci numbers.

Example: Consider the set of Fibonacci numbers {3, 5, 8, 13}. According to the above formulas $a = 3 \cdot 13 = 39$; $b = 2 \cdot 5 \cdot 8 = 80$, and $c = 5^2 + 8^2 = 89$. Because $39^2 + 80^2 = 89^2$, these numbers form a Pythagorean Triple.

7. Consider the set of Fibonacci numbers {8, 13, 21, 34}. Calculate the values a, b, and c using the above formulas and verify that they form a Pythagorean Triple.

8. Any two positive integers can generate a Fibonacci pattern different from the actual Fibonacci sequence. Pick any two positive integers and call them F_1 and F_2. Let $F_3 = F_1 + F_2$ and $F_4 = F_2 + F_3$. Use these four values to calculate the values a, b, and c using the above formulas and determine if they form a Pythagorean Triple.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

1. Prove that at least one number in a Pythagorean Triple is even.

2. Prove that at least one of the numbers in every Pythagorean Triple generated by $m^2 - n^2$, $2mn$, $m^2 + n^2$ is divisible by 3, at least one is divisible by 4, and at least one is divisible by 5.

3. Find a Pythagorean Triple $\{a, b, c\}$ that is not formed by $a = m^2 - n^2$; $b = 2mn$; and $m^2 + n^2$. Explain your method.

4. Prove that if $F_3 = F_1 + F_2$ and $F_4 = F_2 + F_3$, then $F_3^2 - F_2^2 = F_1 \cdot F_4$.

Additional Resources

- This book contains dynamic geometry sketches of many proofs of the Pythagorean Theorem. Pythagoras Plugged In: Proofs and Problems for the Geometer's Sketchpad, Dan Bennett, Key Curriculum Press, 1995.

- For more information and perspectives on Pythagorean Triples, see: **http://mathworld.wolfram.com/PythagoreanTriple.html**.

- For more information and proofs about the properties of Pythagorean Triples, see: **http://www.cut-the-knot.org/pythagoras/pythTriple.shtml**.

- For a Java applet that generates Pythagorean Triples, see: **http://www.ces.clemson.edu/~simms/neat/math/pyth/**.

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