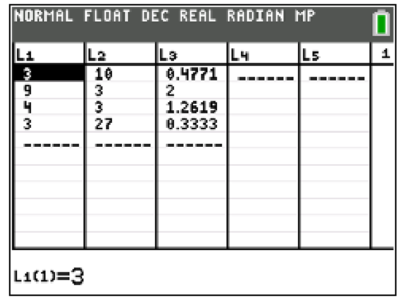


Math Objectives

- Students will discover the change of base rule for logarithms by examining the ration of two logarithmic functions with different bases.
- Students will review and use the properties of logarithms.
- Students will evaluate and simplify logarithmic expressions while using the change of base rule.
- Students will use appropriate tools strategically (CCSS Mathematical Practice).

Vocabulary

- logarithmic function
- change of base
- properties of logarithms

About the Lesson

- This lesson involves manipulating logarithmic functions with different bases.
- As a result, students will:
- Use different properties of logarithms.
- Solve simple equations.
- Derive a formula that allows you to convert a logarithm to a different base..

Teacher Preparation and Notes.

This activity is done with the use of the TI-84 family as an aid to the problems.

Activity Materials

- Compatible TI Technologies: TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE

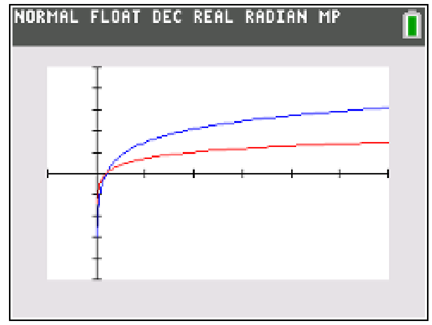
** with the latest operating system (2.55MP) featuring MathPrint™ functionality.*

Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity
Change_of_Base_84CE_Student.pdf
Change_of_Base_84CE_Student.doc
DIFFBASE.8xp



In this activity, students discover the change of base rule for logarithms by examining the ratio of two logarithmic functions with different bases. It begins with a review of the definition of a logarithmic function, as students are challenged to guess the base of two basic logarithmic functions from their graphs. The goal of applying the properties of logarithms to add these two functions is introduced as a motivator for writing them in the same base. Students explore the hypothesis that the two functions are related by a constant first by viewing a table of values, then by exploring different values for the two bases. Finally, they prove the change of base rule algebraically and apply it to find the sum of the two original functions.

Teacher Tip: If you are using the *DIFFBASE.8xp* file, you will need to practice a little with running the program. After downloading the program onto the handheld, you will press **PRGM**, **1: TI-Basic**, **DIFFBASE**. Practice going through the 4 menu options. To leave each option and go back to the menu, press **ENTER**. You can then select again or press 5 to exit. You will need to exit to see the table of values.

Problem 1 – Relating log functions with different bases

Execute the **DIFFBASE** program. Press **PRGM** and choose it from the list. Press **ENTER**. Choose **SeeGraphs** from the menu. You will see the graphs of two logarithmic functions with different bases:

$$\boldsymbol{Y_1 = \log_a(x)} \text{ and } \boldsymbol{Y_2 = \log_b(x)}.$$

(a) What are *a* and *b*? Approximate the values know that the x-scale is every 5 units and the y-scale is every 1 unit.

<u>Solution:</u> Students should estimate the values with their cursor to find a and b. $\boldsymbol{a = 3}$ and $\boldsymbol{b = 10}$.

(b) What points on the graph are the best clues to the base of the logarithmic function?

<u>Solution:</u> Students should realize that the most informative points on the graphs will be those at which y = 1 or some other whole number. When y = 1, x is equal to the base of the logarithm.

Once you think you know you *a* and *b*, run the **DIFFBASE** program again and choose **GuessBases** from the menu. Enter the values of *a* and *b* that you found.

The program graphs two logarithmic functions with bases you entered as thick lines on top of the original graph. If you choose *a* and *b* correctly, your graph will look like the one shown.

If you see more than 2 curves on your graph, try different values for *a* and *b*.

Suppose we are interested in the sum of these two functions,

$$(Y_1 + Y_2)(x) = \log_a(x) + \log_b(x).$$

How could we write this as a single logarithmic expression?

- We can't apply the properties of logarithms unless the logarithms have the same base.
- We need to rewrite the functions with the same base.
- This means we want to find a function that is equal to Y_1, but has a log base ***b*** instead of log base ***a***.

Maybe there is a constant *c* that could relate the two functions, like: $c \cdot Y_1(x) = Y_2(x)$. Then, we would have $Y_1(x) = \frac{Y_2(x)}{c} = \frac{1}{c} \cdot \log_b(x)$, which is a logarithmic function with base *b*.

We can't be sure there is such a constant, but that doesn't have to stop us from looking for one. Run the **DIFFBASE** program again and choose **GraphC** from the menu. The program calculates $c = \frac{Y_2(x)}{Y_1(x)} = \frac{\log_b x}{\log_a x}$ and stores the result in Y_3.

Examine the graph of **Y3** and then view the **Y3** function table.

(c) What is ***c***?

Solution: c = 0.47712

Problem 2 – A closer look at *c*

Is *c* always the same? Run the **DIFFBASE** program and choose **CalculateC** from the menu. Given *a* and *b*, the program calculates ***c*** and displays the value. Try two different values of *a* and *b*. What is *c* now?

Continue to choose **CalculateC** from the menu to experiment with different values of *a* and *b*. As you try different values, the program records the results in the **Lists**. (Values of *a* are stored in **L1**, *b* values in **L2**, and *c* values in **L3**.) Be sure to try some powers of *a* and *b* such that one is a power of the other, like 2 and 8 or 9 and 3.

After you have tried at least 10 different values for *a* and *b*, exit the program and view the data in the lists. Record some of those values in the table below.

(d) Can you guess a formula for *c*?

Sample Table:

a	**f1(*x*)**	***b***	**f2(*x*)**	***c***
3	$\log_3 x$	**27**	$\log_{27} x$	**0.33333**
6	$\log_6 x$	**36**	$\log_{36} x$	**0.5**
2	$\log_2 x$	**16**	$\log_{16} x$	**0.25**
9	$\log_9 x$	**3**	$\log_3 x$	**2**

Formula: $\boldsymbol{c = \frac{1}{\log_a b}}$

Teacher Tip: In Problem 2, students will test different values of a and b (the bases of the logarithmic functions) to see how they affect the value of c. This creates a function table that gives students an intuitive sense of the multivariable function c(a, b). Guide students to guess a rule for c based on this data. If you wish, direct them to calculate 1/c in Column D as a way of giving a hint. $\left(c = \frac{1}{\log_a b}\right)$.

Problem 3 – Deriving the Change of Base Rule algebraically

We are convinced now that there is a constant that relates $\mathbf{log}_{\boldsymbol{a}}(\boldsymbol{x})$ to $\mathbf{log}_{\boldsymbol{b}}(\boldsymbol{x})$ and that the constant depends on the values of ***a*** and ***b***. We may even have an idea what the constant is. Time to use some algebra to find out for sure.

Two functions are equal if and only if their values are equal for every *x*-value in their domain. Let's pick a point (*x*, *y*) on the graph of $\boldsymbol{Y_1(x)}$. For this (*x*, *y*), $\mathbf{log}_{\boldsymbol{a}}(\boldsymbol{x}) = \boldsymbol{y}$. If we can write ***y*** in terms of logs base ***b***, we will have our function.

(e) Rewrite $\mathbf{log}_{\boldsymbol{a}}(\boldsymbol{x}) = \boldsymbol{y}$ as an exponential function.

Solution: $a^y = x$

(f) We want an expression with base ***b*** log, so take $\log_b$ of both sides.

Solution: $\log_b a^y = \log_b x$

(g) Simplify using the properties of logs. Solve for ***y***.

Solution: $y \cdot \log_b a = \log_b x$

$$y = \frac{\log_b x}{\log_b a}$$

(h) What is ***c***?

Solution: $c = \frac{1}{\log_a b}$

You have found a formula for changing the base of a logarithm. To change a log base ***a*** expression to log base ***b***, simply divide the expression by $\mathbf{log}_{\boldsymbol{a}}(\boldsymbol{b})$. This can be written as

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}.$$

(i) Use this formula to find $(\boldsymbol{Y_1} + \boldsymbol{Y_2})(\boldsymbol{x})$ if $\boldsymbol{Y_1}(\boldsymbol{x}) = \mathbf{log_3}(\boldsymbol{x})$ and $\boldsymbol{Y_2}(\boldsymbol{x}) = \mathbf{log_{10}}(\boldsymbol{x})$.

Solution:
$$\begin{aligned}(Y_1 + Y_2)(x) &= \log_3 x + \log_{10} x \\ &= \frac{\log_{10} x}{\log_{10} 3} + \log_{10} x \\ &= \frac{1}{\log_{10} 3}(\log_{10} x + \log_{10} 3 \cdot \log_{10} x) \\ &= \frac{1}{\log_{10} 3}\left(\log_{10} x + \log_{10} x^{\log_{10} 3}\right) \\ &= \frac{\log_{10} x^{1 + \log_{10} 3}}{\log_{10} 3}\end{aligned}$$

Teacher Tip: Problem 3 steps students through the process of deriving the Change of Base rule algebraically. You may wish to have students record their work on a piece of paper instead of typing their expressions into the handheld.

Problem 4 – Further practice with the Change of Base Rule

(j) Use the Change of Base Rule to simplify the expression: $\frac{1}{4}\log_9 27$.

Solution: $\frac{1}{4}\log_9 27 = \frac{1}{4} \cdot \frac{\log_3 27}{\log_3 9} = \frac{1}{4} \cdot \frac{3}{2} = \frac{3}{8}$

(k) The function f is given by $f(x) = \log_4(x)$. Without a handheld, use the Change of Base Rule to evaluate $f(8)$.

Solution: $f(8) = \log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2}$

(l) If $f(x) = \log_8 x$ and $g(x) = \log_{64} x$, given that the input values into each function are equal, describe the relationship between the output values of $f(x)$ and $g(x)$.

Solution: Using the Change of Base Rule, the outputs of g(x) will be one half the outputs of f(x).

$$\mathbf{log}_{64}\, x = \frac{\mathbf{log}_8\, x}{\mathbf{log}_8\, 64} = \frac{f(x)}{2}$$