Modelling Parabolas – The Bridge

Student Activity – Teacher Notes

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TI-Nspire™ Investigation Student 50 min

Teacher Notes:

It is important that students understand the idea of modelling, including the benefits and the limitations. We can use models to make calculations or predictions within the domain of the model, it does not necessarily mean the material is built around that model. The Walkie Talkie building in London is a perfect example. From a practical perspective, it is irrelevant whether the architects and engineers designed the shape of the building around a parabola. The faced of the building can be modelled by a parabola and therefore elicits some of the associated characteristics, in this case, a focal point that created a region of intense heat at various times of the day and year. The Capricorn Caves in Queensland represent a natural formation that provides amazing acoustics. The light and dark regions in the bottom of a pool are created by the waves on the surface focusing the light. The curvature of the waves can, in part, be modelled by a parabola.

In this activity students model the curvature of the Sydney Harbour bridge, that doesn't mean the curvature of the bridge is parabolic. The lower arch can be modelled remarkably well by a parabola, however the upper arch is better suited to a quartic function (quartic function not required for this activity), over the domain: tower to tower.

Another important aspect of the activity is for students to understand how the cartesian plane is used as a reference system. The idea that the origin can be placed at a point that is most useful, in this case to take advantage of the symmetry of the bridge.

Australian Curriculum Standards

AC9M9A02

Simplify algebraic expressions, expand binomial products and factorise monic quadratic expressions.

AC9M9A04

Identify and graph quadratic functions, solve quadratic equations graphically and numerically, and solve monic quadratic equations with integer roots algebraically, using graphing software and digital tools as appropriate.

AC9M9A06

Investigate transformations of the parabola $y = x^2$ in the Cartesian plane using digital tools to determine the relationship between graphical and algebraic representations of quadratic functions, including the completed square form, for example: $y = x^2 \rightarrow \; y = \frac{1}{3}x^2$ (vertical compression) …

AC9M10A05

Experiment with functions and relations using digital tools, making and testing conjectures and generalising emerging patterns.

Author: P. Fox

Introduction

The building located at 20 Fenchurch St London, often referred to as the “Walkie Talkie” building provides an example of the importance of modelling. It is unlikely the building’s curvature was designed specifically around a parabola, however, the fact that it can be modelled by one means it possess some parabolic properties. One of these properties (focal point) caused significant issues leading to its other nickname: “The Fry Scraper”.

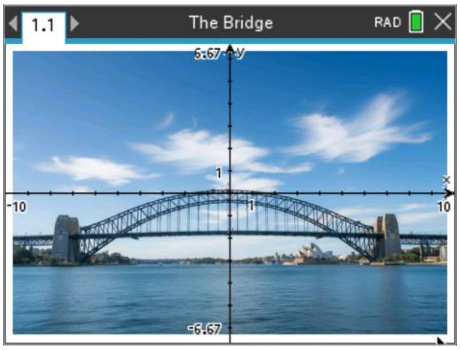
Much debate exists around the iconic Sydney Harbour bridge, is it a parabola or a catenary. The purpose of this activity is to ‘model’ the bridge using a parabola. The distinction here is that we are modelling the curve, not providing any theoretical proof of its nature. Understanding the purpose of a model, it’s opportunities and limitations is a useful mathematical exercise in its own right.

Set up

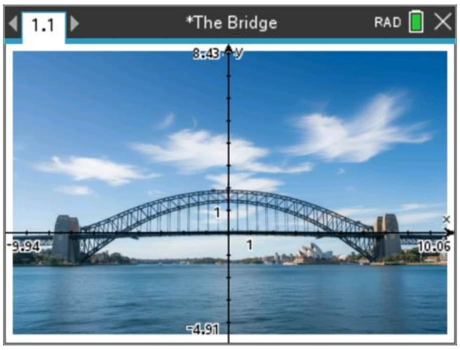
Open the TI-Nspire document: “The Bridge”.

A picture of the bridge has been inserted in the Graphs Application for the purposes of modelling the two main arches and road way.

The axes provide a numerical reference frame for the model. To make the reference frame meaningful it needs to have a common sense reference point and scale.



There are several opportunities for the placement of the origin. For the purposes of this activity, the reference point will be placed as close as possible to the centre of the bridge (laterally) with the x axis aligning to the roadway of the bridge. You can achieve this by changing the window settings directly or by clicking and dragging the axes.



> You may notice that the roadway for the bridge is not perfectly flat (straight), this is done on purpose! Imagine standing in the middle of the bridge. Your weight is pushing down on the roadway. Now imagine the enormous load placed by vehicles. If the road dips under all the weight, the curve will flatten out. As the curve flattens out or straightens, it pushes outwards. For this to happen the pillars (bridge supports) would need to move further apart. The pillars are enormous and will not move, the roadway for the bridge is therefore in compression, making it even stronger.

Author: P. Fox

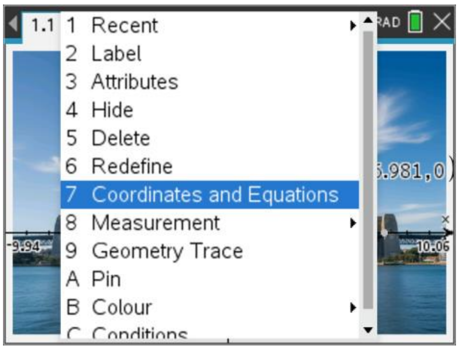
Key Measurements:

- Distance between granite faced pylons = 503m
- Highest point on bridge above roadway = 73m

To scale the window and align the cartesian plane with the actual distance measurements, a point can be placed on the x axis.

- Press: **P** to create a Point.
- Place the point on the x axis.
- Measure the coordinates. (ctrl + menu on the point)
- Move the point(s) to align with the granite faced pillars.

Note: Natural perspective makes the second (rear) pillar visible. Align the point with the front pillar.



You can change the colour and style of the point to make it easier to see.

- Changing the attributes of the point allows you to make it appear as a 'cross'.
- Changing the colour to white makes it more visible.

Question: 1

Scaling the Window:

a) Move the point to align with the inside edge of each of the two pillars and record the respective coordinates.

Answer: Answers will vary slightly. Depending on where students have located the origin, the ordinate for the pillars should be approximately: ⁻7.2 ~ ⁻6.9 for the left pillar and 6.9 ~ 7.1 for the right pillar.

b) Using the "distance between pylons" and previous measurements, determine the corresponding scale factor.

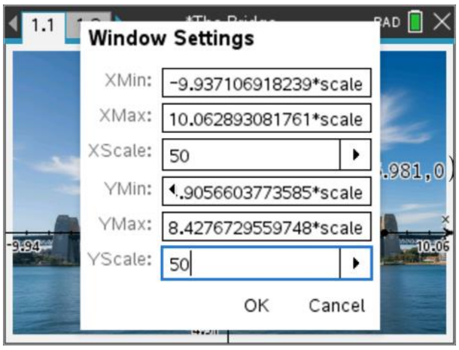
Answer: Actual distance between the pillars: 503m. Current cartesian measurements (approx) 14. Students may express the scale factor as: 14 : 503 (or reverse). Students should be encouraged to store it as a numerical quantity such as: 503/14 ≈ 35.9. Typical range of scale factor 35 ~ 36.5

The scale factor can be stored to make it easier to use.

The window settings can be adjusted using the stored variable. In the example shown here, the scale factor has been stored as: "scale".

Notice that the "xscale" in the Window Settings has been changed to an appropriate value. (Hash marks every 50 metres).

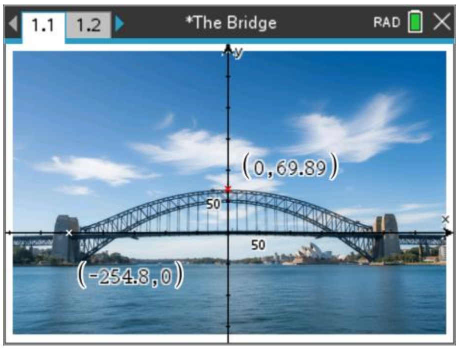
Notice also that the y scale must also be adjusted by the same scale factor to ensure the window remains 'square'.



The image of the bridge is approximately symmetrical about the y axis. Place a new point on the y-axis and capture the coordinates of the point as close as possible to the top of the bridge.

The graph shown opposite illustrates this and captures the height of the bridge. The measurement: 69.9 which is quite close to the actual measurement of 73m, and certainly within the margin of error given the respective scales.

This confirmation means we are ready to model the curvature of the bridge and that the axis have been set correctly.



Modelling the lower arch:

There are two arches on the bridge the 'upper' and 'lower. We start with the lower arch.

Question: 2 Equation 1 – Difference of perfect squares

a) Move the point on the y – axis to the top of the lower arch and record the coordinates.

Answer: Answers will vary ≈ (0, 51)

b) Move the point on the x – axis to where the lower arch intersects the x axis. This point may be slightly higher than the actual roadway and slightly different on either side. For the purposes of this model we assume that the bridge is symmetrically oriented around the y – axis. Record the coordinates of the axis intercepts.

Answer: Answers will vary ≈ (-175, 0) [Left] and (173, 0) [Right]. Students may use one or the other intercept for the equation or they may choose to *average* them.

c) Use the previous two results to determine an appropriate equation of the form: $y = a(x-m)(x+n)$.

Answer:

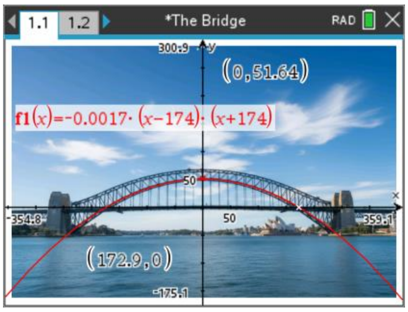
Part B provides m and n directly.

Example: $y = a(x-174)(x+174)$

Substitute $x = 0$ and the coordinates from Part (a) to establish $a \approx 51 \div (-174 \times 174) \approx -0.00168$

Sample Graph:

Comments: The graph provides are reasonable fit!



You may need to hide your equation to ensure maximum visibility for the next question.

Question: 3 Equation 2 – Translational Form:

a) You no longer have to assume the image is symmetrical. Determine the coordinates for the x axis intercepts.

Answer: Students can use their answers from Question 2.

b) Use your previous result to help locate the coordinates for the top of the lower arch.

Answer: If students continue with the notion that the bridge as symmetrically oriented around the y axis, their answer will be the same as before. If students use say: (-175, 0) and (173, 0) as the points where the arch crosses the axis, they should use the location of the turning point as (-1, #). This reflects the pure mathematics approach, from a modelling perspective, students will not notice any difference in the 'accuracy' of the model.

c) Use your results from part (b) to determine the values of h and k, and then your results from part (a) to determine the corresponding value for a for the equation of the form: $y = a(x-h)^2 + k$.

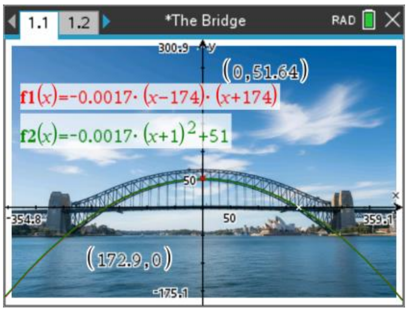
Answer: Students must use the coordinates from their chosen turning point for h and k. Determination of the value for a, found by substitution of $(x_1, 0)$ or $(x_2, 0)$ (intercepts).

Example: $y = a(x+1)^2 + 51$ … substitute (-175, 0) to get

$a \approx -51 \div 174^2 \approx -0.00168$

Sample Graph:

Comments: The graph is almost identical!



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Question: 4 Compare the pair:

a) Compare the methods used to determine the equations in Questions 2 & 3. Which method did you find easier?

Answer: The results should be almost identical. Methods: Students should recognise that both approaches require the dilation to be determined using substitution. In both cases the values obtained from locating either the axis intercepts or the turning point, involve direct substitution.

b) The final equations from Questions 2 & 3 should be very similar given they are modelling the same arch. Compare the two equations. (Hint: Express them both in expanded form.)

Answer: Sample expanded versions: $y \approx 0.00168x^2 + 51$ vs $y \approx -0.00168x^2 - 0.00336x + 51$

Modelling the upper arch:

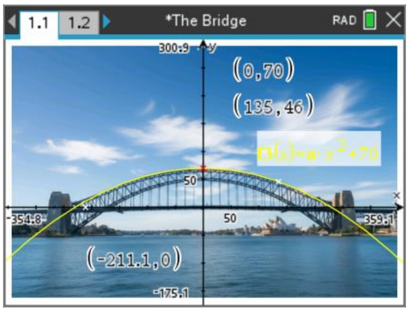
Question: 5 Equation 3 – Upper Arch

Using the same approach as the lower arch, determine an equation for the upper arch and comment on:

a) How well a parabola models the upper arch.

Answer: The easiest option for students to use is the translational form as the upper arch doesn't cross the x axis, the arch would need to be extended to estimate the points accordingly. Students should also note that the upper arch has a slightly different curvature as it nears the granite pillars.

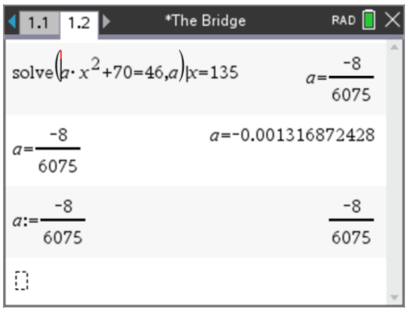
Sample Equation: $y = -0.00132x^2 + 70$



b) Which method(s) you used to determine the equation.

Answer: Most students will use the turning point form otherwise they need to 'predict' where the upper arch will intersect the x axis. Another, more elaborate approach would be to insert a horizontal line at say y = 20 and then using the free floating point to identify where the upper arch passes through this point and the translate the function accordingly.

Example: $y = -0.00132(x - 195)(x + 195) + 20$



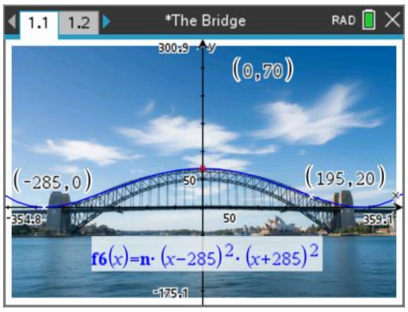
c) Adaptations to the equation to make it a better 'general' fit.

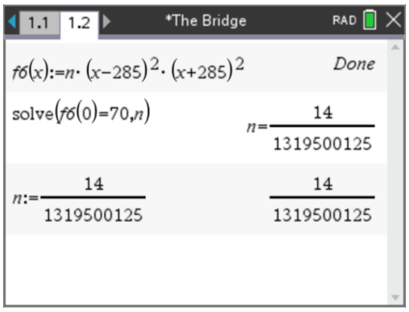
Answer: Answers will vary. Student may try and accommodate for the upturn (regions adjacent to the granite covered pillars), but will result in a worse fit for the majority of the arch.

Domain restrictions could be referenced.

Modelling as a combination of three parabolas (piecewise function) but that faces challenges of ensuring the equations connect, and connect 'smoothly'.

Students could use a quartic function … 'extension' … some of the other procedures are the same.





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Modelling the roadway & Water:

Question: 6 Equation 4 – Roadway:

You may have noticed the roadway is not perfectly flat. This is done on purpose!

a) The roadway for the bridge is not flat. Use the free-range point and move it to the far right of screen where the road dips below the x - axis. Write down the coordinates of this point.
Answer: Approx: (345, -7)

b) The road can be modelled by the equation: $y = ax^2$. Use your result from the previous question to determine the value of 'a'.

Answer: Using the above coordinates: $a = \frac{-7}{345^2}$ therefore: $y = \frac{-7}{345^2}x^2$

Question: 7 Equation 5 – Sea Level

The sea level is as horizontal as it can be! Determine an equation to model the sea level.

Answer: $y = -50$

Extension: Linking it Together

Question: 8 Cables:

There are 19 clearly visible cables connecting the lower arch to the roadway. The appearance of two adjacent cables is due to the viewing angle. The further away from the middle, the more prominent the double cable appearance.

a) Use a free-range point to determine the location of the first cable (left) connecting the lower arch to the roadway. Record the coordinates of this point.

Answer: Approx: (-145, 0)

b) Determine the distance between successive cables. (Show your working out)
Answer: There are 19 cables. The cable furthest to the right is at approximately (145, 0). The total distance between the extreme (visible) cable points is therefore 290m. There are 19 cables which corresponds to 18 divisions: 290 ÷ 18 ≈ 16m.

c) The length of any one of the cables can be calculated by: $f_2(x) - f_4(x)$ where:

$f_2(x)$ is the equation for the lower arch;

$f_4(x)$ is the equation for the road;

x is the abscissa (x coordinate) of the cable.

Determine the total length of the cables supporting the bridge.

Answers: Measurements are approximate only.

Cable Number	1	2	3	4	5	6	7	8	9	10
Cable Coordinate	-145	-128.9	-112.8	-96.7	-80.6	-64.4	-48.3	-32.2	-16.1	0
Cable Length:	19.3	26.4	32.6	37.9	42.4	46.1	48.8	50.7	51.8	51.9
Cable Number	11	12	13	14	15	16	17	18	19	Total
Cable Coordinate	16.1	-32.2	48.3	64.4	80.6	96.7	112.8	128.9	145	
Cable Length:	51.8	50.7	48.8	46.1	42.4	37.9	32.6	26.4	19.3	741.5

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d) Jess approximated the total length (sum) of the cables using the following formula:

$$\sum_{n=0}^{18}\left(f_2(-146+16n)-f_4(-146+16n)\right)$$

Explain how this formula works. (Test it on your own calculations where -146 is the starting point of the first cable on the left.

Answer: When n = 0 (first value), calculation = $f_2(-146) - f_4(-146)$, which is the first cable length.

When n = 1, (second value), calculation = $f_2(-146+16.1) - f_4(-146+16.1)$, next cable length.

This process is repeated for all cable lengths. Total cable length, using the functions defined for the previous questions: 741.8m. Calculating manually results in the same value, within the corresponding rounding errors. (741.5m)

Question: 9 The Bridge Climb:

Yes, this is a thing!

According to the National Construction Code (NCC), for industrial and commercial buildings specifically, the Australian Standard AS1657 specifies a maximum riser height of 190 mm.

a) If the external component of the bridge climb started at road level at the base of the granite tower and finished at the top of the top arch, what would be the minimum number of steps required to complete the journey to the top of the bridge?

Answer: Height of bridge above road base ≈ 70m. Each step (max riser) = .19m ∴ 369 steps

b) In addition to the maximum riser height, each step needs a minimum tread length between 240mm and 355mm. Using your previous answer, what is the range of distances potentially covered by the steps? Note any assumptions made in the calculation of this range of lengths.

Answer: 369 steps × 0.24m = 88m (distance is okay). 130m (distance is okay)

c) Discuss any issues that might occur with the steps at the start of the climb. Include calculations to support any issues.

Answer: At the start of the climb the curvature is quite steep.
The location of the first step could be at x = -236m, this is the point ≈ (-236,0.33).
The next step could be at x = -236 + 0.355m, (-236 + 0.355, 0.544)
The rise is approximately 0.21m, however steps are only permitted to be 0.19m each.
Using the small tread the riser only needs to be 0.144m
Therefore, steps at the start of the climb will need a shorter tread than the full 0.355m. The shorter tread would likely lead to more people tripping over.

Author: P. Fox