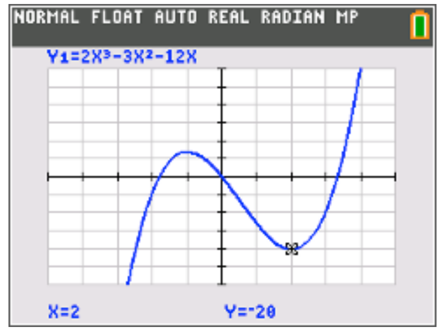
Extrema and Concavity

About the Lesson

In this activity, students will use the first and second derivatives of functions to determine local maximums, minimums, and inflections points. Students will confirm their results graphically and using built-in functions of the graphing calculator. As a result, students will:

- Identify conditions on $f'(x)$ and $f''(x)$ for a local maximum, minimum, and point of inflection to occur.



Vocabulary

- inflection point
- concavity
- extrema

Teacher Preparation and Notes

- This investigation uses **fMin** and **fMax to** verify answers and students will have to restrict the domain when using those commands.
- The calculator should be set in radian mode before the trigonometric derivatives are taken.
- Be sure that the students read the sign rules for minima and maxima in this problem.

Activity Materials

- Compatible TI Technologies:

  TI-84 Plus*
  TI-84 Plus Silver Edition*
  TI-84 Plus C Silver Edition
  TI-84 Plus CE

** with the latest operating system (2.55MP) featuring MathPrint™ functionality.*

Tech Tips:

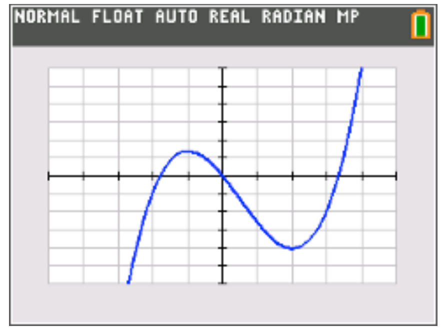
- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials
- Any required calculator files can be distributed to students via handheld-to-handheld transfer.

Lesson Files:

- Extrema_and_Concavity_Student.pdf
- Extrema_and_Concavity_Student.doc

Problem 1 – The Extrema of $y = 2x^3 – 3x^2 – 12x$

Students are to graph the function $y = 2x^3 – 3x^2 – 12x$.



1. Graph the function $y = 2x^3 – 3x^2 – 12x$. Use [–5, 5] for the x dimensions and [–30, 30] for the y dimensions.

a. How many local maximums do you see? Local minimums?

Answer: One local minimum and one local maximum.

b. What is the point of inflection?

Answer: It appears to occur when x = 0.5.

2. Find the first derivative of the function $y = 2x^3 – 3x^2 – 12x$. Set this function equal to zero and solve.

a. What are your solutions?

Answers:

$y' = 6x^2 - 6x - 12 = 0$;

$$6x^2 - 6x - 12 = 0$$
$$6(x^2 - x - 2) = 0$$
$$6(x-2)(x+1) = 0$$
$$x = 2 \text{ or } x = -1$$

b. What is the name given to these solutions?

Answer: critical points

3. Find the second derivative of the original function (or the derivative of the first derivative). Evaluate each of the critical numbers in the second derivative.

a. What are these values?

Answers: The second derivative can be found by taking the derivative of $6x^2 - 6x - 12$.

$y' = 6x^2 - 6x - 12$
$y'' = 12x - 6$

$y''(2) = 12(2) - 6 = 18$

$y''(-1) = 12(-1) - 6 = -18$

b. What would the value be called if the value is positive? Negative?

Answer: If the second derivative of the given *x*-value is positive then that *x*-value yields a minimum. If the second derivative of the given *x*-value is positive then that *x*-value yields a minimum.

c. What is the point of inflection?

Answer: Look at when *y"* is zero.

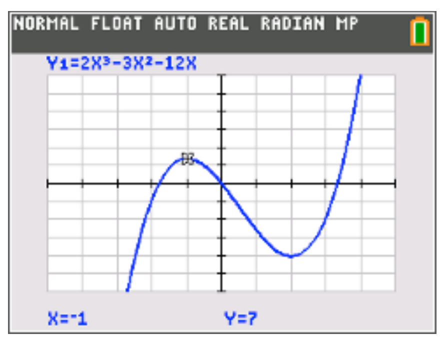
$12x - 6 = 0$
$x = 0.5$

So $x = 0.5$ yields a point of inflection

d. According to your graph, does the function change concavity there?

Answer: Yes, the graph changes from concave-down to concave-up.

4. Use the **trace** command to approach $x = -1$. Look at the *y*-values on both sides of $x = -1$. Do the same for $x = 2$.

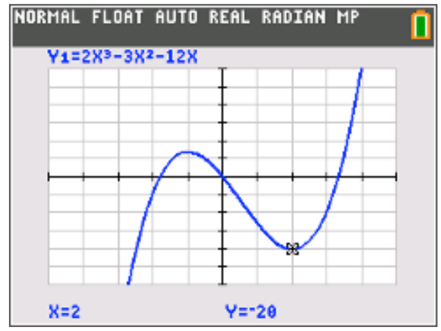


a. Discuss what happens to the y-values on each side of $x = -1$.

Answer: Students should see that the two function values on both sides of $x = -1$ are less than the function value at $x = -1$, making the point a local maximum.

b. Discuss what happens to the *y*-values on each side of $x = 2$.

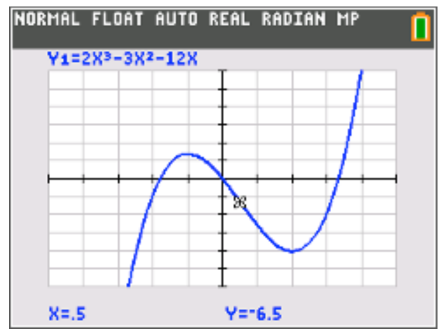
Answer: Students will use the same procedure around $x = 2$ to show that the *x*-values around $x = 2$ yields a greater function value than the value at $x = 2$, making the point a local minimum.



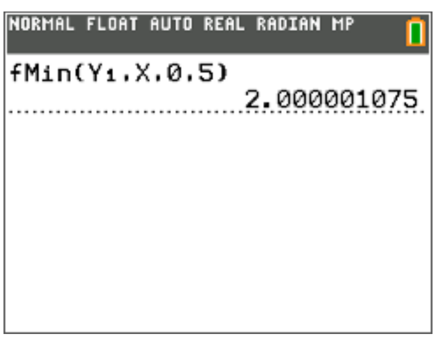
c. What does this discussion tell you about the extrema of the function?

Answer: Extrema will perhaps occur when the derivative of a function is zero.

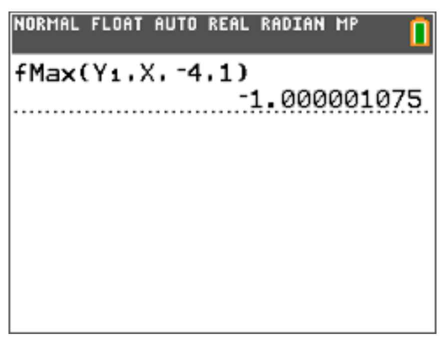
Students are to use the **fMin** and **fMax** commands to verify the maximum and minimum.



You may need to discuss why the value is not exactly 2, due to internal rounding in the calculator.



You may need to discuss why the value is not exactly –1, due to internal rounding in the calculator.

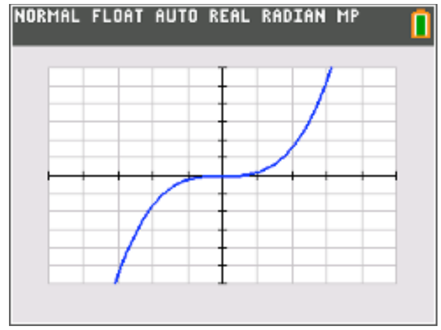


5. Does the use of the **fMin** and **fMax** command yield a similar result?

Answer: yes

Problem 2 – Determining the Extrema of $y = x^3$

The student screen should look like the one at the right.



There are no extrema to be seen on the graph. However, the function changes concavity at $x = 0$. Thus we have a point of inflection but no extrema.

The first derivative is $3x^2$.

$3x^2 = 0$ means that $x = 0$. Therefore, there is a point of inflection when $x = 0$, at the origin.

6. Are there any extrema? If so, at what x-values?

Answer: There are no extrema to be seen on the graph.

7. When does the function change concavity?

Answer: The function changes concavity at $x = 0$. Thus we have a point of inflection but no extrema.

8. What are the critical points?

Answer: The first derivative is $3x^2$. $3x^2 = 0$ means that $x = 0$. Therefore, there is a critical number when $x = 0$, at the origin.

9. What is the point of inflection? Why?

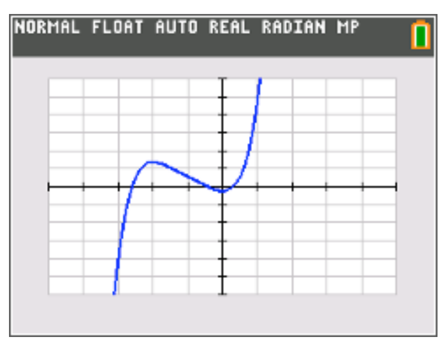
Answer: The second derivative is $6x$. $6x = 0$ means that $x = 0$. Therefore there is an inflection point when $x = 0$, at the origin.

10. If there is no extrema, what interval will **fMin** and **fMax** depend on?

Answer: $(-\infty,\infty)$

Problem 3 – Extrema for Other Functions

Students will now graph $g(x) = (x + 1)^5 – 5x – 2$. They will need to adjust the window settings. Students will use the graph and the **fMin** and **fMax** command to find the extrema.

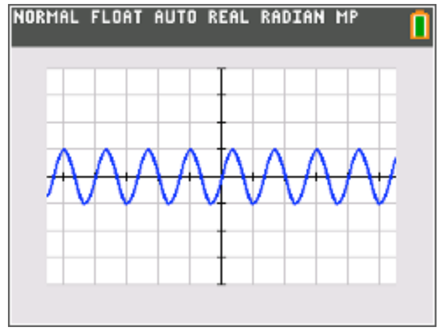


Minimum occurs at (0, –1)

Maximum occurs at (–2, 7)

Inflection Point occurs at (–1, 3)

Graph $h(x) = \sin(3x)$. There will be multiple minima, multiple maxima, and multiple points of inflection.

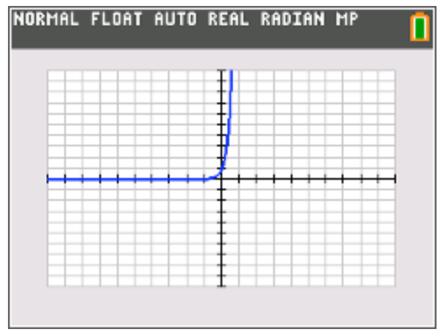


Minimum value of y = –1 occurs at: $x = \frac{\pi}{2} \pm \frac{2n\pi}{3}$.

Maximum value of y = 1 occurs at: $x = \frac{\pi}{6} \pm \frac{2n\pi}{3}$.

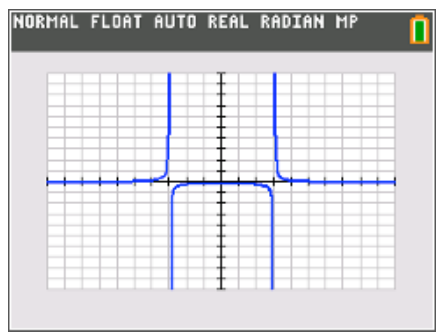
Inflection Points: $x = \frac{n\pi}{3} - 1$; $y = 0$

Graph $j(x) = e^{4x}$. There are no extrema.



$j'(x) = e^{4x} > 0$, so there are no critical points. Also, the function is always concave up so there are no points of inflection

Graph $k(x) = \frac{1}{x^2 - 9}$.



Note that there is a maximum at $x = 0$ but no minimum. There are two vertical asymptotes ($x = 3$, $x = -3$) where the function is not defined. There are no points of inflection because the function changes concavity at the asymptotes.

Remind students that local minimum and local maximum mean just that. The values around $x = 0$ are smaller than at $x = 0$ but all of them are smaller than the values of the function when $x > 3$ or $x < -3$.

It would be a good idea to remind students to turn **On** the "Detect Asymptotes" command found by pressing [2nd] [zoom] (**format**).

11. Graph the following functions. You will need to adjust the window. Find the critical points. Use **trace** to verify the extrema. Then use **fMin** and **fMax** to make a second verification.

<u>Answers:</u>

Function	$g(x)=(x+1)^5-5x-2$	$h(x)=\sin(3x)$	$j(x)=e^{4x}$	$k(x)=\dfrac{1}{x^2-9}$
1st Derivative	$5(x+1)^4-5$	$3\cos(3x)$	$4e^{4x}$	$k(x)=\dfrac{-2x}{(x^2-9)^3}$
2nd Derivative	$20(x+1)^3$	$-9\sin(3x)$	$16e^{4x}$	$k(x)=\dfrac{10x^2+18}{(x^2-9)^4}$
Critical Points	$x = -2$ $x = 0$	$x=\dfrac{\pi}{2}\pm\dfrac{2n\pi}{3}$ $x=\dfrac{\pi}{6}\pm\dfrac{2n\pi}{3}$	none	$x=\pm 3$
Minimum/ Maximum	Minimum: (0, –1) Maximum: (–2, 7)	Minimum value of $y = -1$ occurs at: $x=\dfrac{\pi}{2}\pm\dfrac{2n\pi}{3}$ Maximum value of $y = 1$ occurs at: $x=\dfrac{\pi}{6}\pm\dfrac{2n\pi}{3}$	none	Maximum: $\left(0,\dfrac{-1}{9}\right)$ Minimum: none
Point of Inflection	(–1, 3)	$\left(\dfrac{n\pi}{3}-1, 0\right)$	none	none