EXPLORATIONS

# Chapter 4

## Projectile Motion

In this chapter, you will use the TI-86 differential equation graphing mode to model projectile motion. You will start by analyzing actual data for a falling object with the statistical features of the TI-86. Then you will see how well theoretical differential equations match the actual data. You will also use the differential equation graphing mode to model baseball trajectories with and without air resistance.

## Example 1: Modeling the Motion of Falling Objects with Scatter Plots and Differential Equations

To begin the analysis of projectiles, we dropped a book from a height of 0.8649 meters and measured its height above ground with a Texas Instruments Calculator-Based Laboratory ™ (CBL™). Table 4.1 contains the data for height versus time.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Time** | .00 | .02 | .04 | .06 | .08 | .10 | .12 | .14 | .16 | .18 | .20 | .22. | 24 | .26 | .28 |
| **Height** | .8649 | .8583 | .8484 | .8364 | .8188 | .7991 | .7749 | .7475 | .7156 | .6805 | .6421 | .6004 | .5543 | .5060 | .4544 |

**Table 4.1**

If you can visualize the data, you may gain a better understanding of the relationship between height and time. A common way to visualize the data is to treat each time/height pair as an ordered pair and graph the pairs in a scatter plot. You will compare the scatter plot with the solution to a differential equation, so before you begin, make sure that your graphing calculator is in differential equation graphing mode, clear the differential equation editor (**Q'(t) =** screen), and select **FldOff** in the graph format screen.

### *Solution*

You can create the scatter plot with these steps:

**Enter the data in the list editor**

1. Press [2nd] [STAT] [F2] (**EDIT**) to display the list editor. (Figure 4.1)

   If your list editor is not empty like Figure 4.1, you can remove lists from the editor by using the cursor movement keys to highlight the list name at the top of a list, then pressing [DEL]. This will remove the list from the editor, but not from the TI-86's memory.

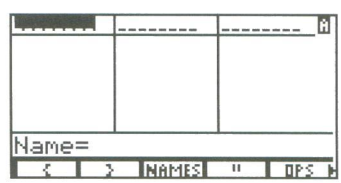


**Figure 4.1**

2. Place the cursor in the box above an empty column. With the cursor here, you can type in the name of a list. The A in the upper right corner of the screen tells us the calculator is in **ALPHA** mode so when you press a key with a letter of the alphabet printed above the key, that letter will be entered on the screen. For example, when you press [−], the calculator enters **T**. Enter letters for the name **TIME**. As you enter letters, you should see them appear at the bottom of the screen on the edit line. (Figure 4.2)

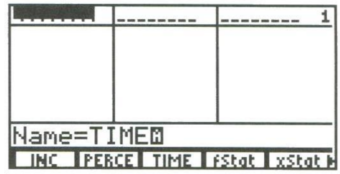


**Figure 4.2**

3. Press [ENTER] and the name should appear at the top of the column. (Figure 4.3)

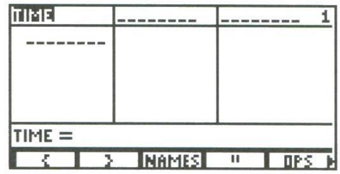


**Figure 4.3**

4. Press [▼] to move the cursor down to the first empty line below the list name. Now you can enter the time data from Table 4.1 on the previous page. Notice that as you type in a value, it appears on the edit line at the bottom of the screen. (Figure 4.4)

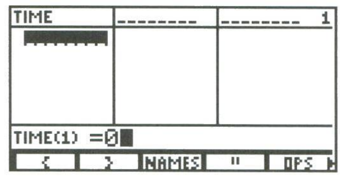


**Figure 4.4**

5. When you press [ENTER], the value is placed in the column and the cursor moves to the next row. (Figure 4.5)

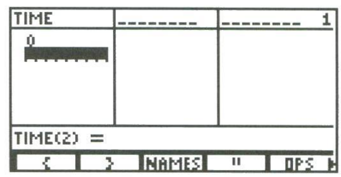


**Figure 4.5**

6. Enter the remaining time values. The cursor should scroll down when you reach the bottom of the screen. After you have entered all of the values, your screen should look like Figure 4.6.

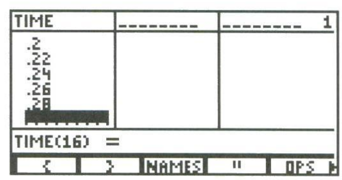


**Figure 4.6**

7. Use the cursor movement keys to move back to the top of the column until **TIME** is highlighted again. As you hold down [▲], the cursor automatically scrolls to the top of the list. (Figure 4.7)

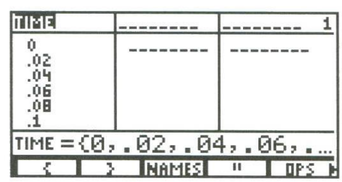


**Figure 4.7**

8. With the cursor at the top of the **TIME** column, you can use the right cursor movement key to move to the top of the next column and enter the name for the heights. Enter the name **HEIGHT** in the column next to **TIME**. Enter the height data in this column. Your screen should look like Figure 4.8 after you enter the last height.

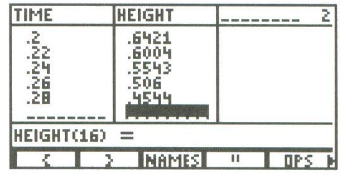


Figure 4.8

**Set up the scatter plot**

1. Press [2nd] [STAT] [F3] **(PLOT)** to set up the scatter plot. (Figure 4.9)


Figure 4.9

2. Press [F1] to select **Plot 1**. (Figure 4.10)

   Some of your settings may not be the same as those shown in Figure 4.10, but you will make corrections in the steps below.


Figure 4.10

3. Use the cursor to highlight **On** and then press [ENTER]. This turns the scatter plot on. Move the cursor to the next row to select the type of plot. Press [F1] (**SCAT**) to select a scatter plot. The icon to the right of **Type=** should indicate a scatter plot. (Figure 4.11)


Figure 4.11

4. Move the cursor to **Xlist Name=**. A submenu containing list names should appear. If there are more than five list names, you can press [MORE] to scroll through the other names. Press the function key corresponding to the name **TIME**. The name **TIME** should now appear beside **Xlist Name=**. After entering the **Xlist Name**, move the cursor to **Ylist Name=** and enter the name **HEIGHT** by pressing [F4] (the function key corresponding to **HEIGHT)**. (Figure 4.12)


Figure 4.12

5. Move the cursor to the last row and press [F1] to select squares as the symbol to be used to plot each point. (The other two symbols that could be used to plot points are + and . .) (Figure 4.13) The scatter plot is now set up.


Figure 4.13

### Select the viewing window and plot the data

1. The Zoom Data feature (**ZDATA**) in the ZOOM menu will automatically select an appropriate viewing window for the scatter plot you have set up. Press [GRAPH] [MORE] [F3] (**ZOOM**) to access the ZOOM menu. (Figure 4.14)


Figure 4.14

2. Press [MORE] [F5] (**ZDATA**) to select the Zoom Data feature. This not only selects an appropriate window, but also graphs the *scatter plot.* (Figure 4.15)

   It is important to realize that this graph does not show the actual path of the falling book. Although the book dropped straight down, the graph shows us a curve because it represents how time and height are related for the falling book.

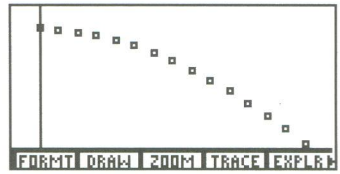


Figure 4.15

Now compare the solution to a differential equation for falling objects with the scatter plot. If you neglect air resistance, the differential equation is

$$\frac{d^2y}{dt^2} = -9.8$$

where $y$ = height, and $t$ = time. The general solution to this differential equation is

$$y = -4.9t^2 + v_o t + s_o,$$

where $v_o$ is the initial velocity and $s_o$ is the initial height.

### *Solution*

1. Enter the system of differential equations for acceleration in the differential equation editor (**Q'(t) =** screen). Deselect **Q2** and select the thick graphing style for **Q'1**. Notice **Plot1** is highlighted at the top of the **Q'(t) =** screen. This tells us **Plot1** (statistics plot of the data) is selected for graphing. (Figure 4.16)


Figure 4.16

2. You can use the same **xMin/xMax** and **yMin/yMax** values you used for the scatter plot, but you need to enter 0 for **tMin**, .28 for **tMax** and .01 for **tStep** in the window editor. (Figure 4.17)

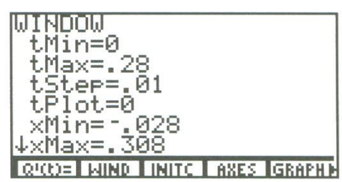


Figure 4.17

3. Make sure the Field is off and the Axes are set to **x = t** and **y = Q1** Enter the initial conditions (Figure 4.18) and draw the graph. (Figure 4.19)

   The solution to the differential equation matches the scatter plot fairly well although it diverges towards the end of the plot.

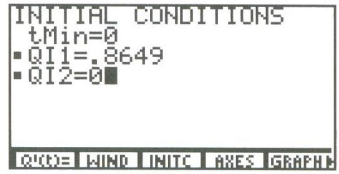


Figure 4.18

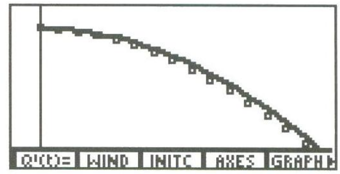


Figure 4.19

Before you leave this example, you will use the TI-86 to find a curve of best fit for this scatter plot and then compare the best fit curve with the analytic solution to the differential equation.

1. Since you will enter the best fit curve and the analytic solution as functions, you need to change to function graphing mode. Press [2nd] [MODE] and select **Func**, then exit to the home screen. (Figure 4.20)


Figure 4.20

2. The analytic solution is

$$y = -4.9x^2 + .8649$$

   where $x$ is time and $y$ is height. Since the analytic solution is a second-degree polynomial, you should look for a second-degree polynomial for the curve of best fit. This curve of best fit is also called a *regression curve*.

   The TI-86 regression curves are calculated from the STAT CALC menu. Press [2nd] [STAT] [F1] (**CALC**) to select this menu then press [MORE] [F4] (**P2Reg**) to paste the P2Regression command to the home screen. (Figure 4.21)


Figure 4.21

3. To enter the names of the $x$ and $y$ lists and designate a place to store the regression equation, press [2nd] [LIST] [F3] **(NAMES)**. You should see a menu containing the list names. Press [F3] (**TIME**), then press [,]. Next press [F1] (**HEIGH**). Then press [,] [2nd] [ALPHA] [Y] [1] to store the regression equation in **y1**. The $y$ in **y1** must be lower case. (Figure 4.22)


Figure 4.22

4. Press [ENTER] and you should see a description of the regression equation that best fits the data. (Figure 4.23)


Figure 4.23

5. The numbers at the bottom of the screen are the coefficients of the second degree polynomial. You can scroll to see all of the coefficients by pressing [▸]. The regression equation rounded to four decimal places is

$$y = -4.4781x^2 - .2155x + .8648\ .$$

How close is this equation to the analytic solution? You will store both equations in the **y(x)=** menu and compare their graphs in order to answer this question.

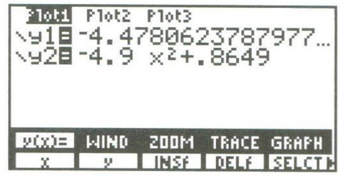


Figure 4.24

6. The regression equation has already been stored in the equation editor (**y(x)=** screen) in **y1**. Press [GRAPH] [F1] to see this menu and then enter the analytic solution in **y2**. (Figure 4.24)

7. You need to reselect an appropriate viewing window since you changed to function mode. Press [2nd] [F3] to select the ZOOM menu then press [MORE] [F5] **(ZDATA)** to see the curves overlaid with the scatter plot. (Figure 4.25)

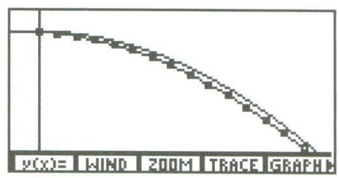


Figure 4.25

The first curve drawn was the regression polynomial (curve of best fit). This polynomial fits the actual data for the dropping book. The second curve drawn was the analytic solution to the differential equation. This curve is essentially the same curve drawn by the TI-86 in differential equation mode in Figure 4.19. It represents the theoretical model for the dropping book. The theoretical model is close but not exactly the same as the regression polynomial. The middle term in the regression polynomial is the initial velocity. This term suggests the ball may have had an initial velocity that differed slightly from zero. This would help explain the difference between the two curves since the theoretical solution assumed an initial velocity of zero.

The projectile (book) in Example 1 moved in only one dimension. In Example 2, you will model the motion of a projectile moving in two dimensions.

## Example 2: Modeling Baseball Trajectories

A baseball is hit when it is 3 feet above the ground. It leaves the bat with an initial velocity of 152 feet per second at an angle of $20^\circ$ with the horizontal. How far will the ball travel?

### *Solution*

The baseball moves both horizontally and vertically. If you neglect air resistance the horizontal acceleration is $x'' = 0$ and the vertical acceleration is $y'' = -32$ ft/sec$^2$. You can use the following systems to represent these differential equations:

horizontal distance = $x = Q1$

horizontal velocity = $x' = Q'1 = Q2$

horizontal acceleration = $x'' = Q''1 = Q'2 = 0$

height = $y = Q3$

vertical velocity = $y' = Q'3 = Q4$

vertical acceleration = $y'' = Q''3 = Q'4 = -32$

1. Select **DifEq** graphing mode and enter these systems in the differential equation editor. Use the **STYLE** feature to select the Animate/Path graphing style for **Q'1** and **Q'3**. If your scatter plot is still selected from the last example, deselect it by moving the cursor to highlight **Plot1** and then press [ENTER]. (Figure 4.26)


Figure 4.26

2. If you estimate that the ball would be in the air for under five seconds, then enter 0 for **tMin** and 5 for **tMax** in the window editor. Enter .1 for **tStep** in order to see the ball every tenth of a second. Then enter a [0, 500] x [⁻15, 100] viewing window with **xScl** = 50 and **yScl** = 10. (Figures 4.27 and 4.28)

   The initial horizontal distance is zero and the initial height is 3. The initial velocity is 152 ft/sec. This velocity can be resolved into horizontal and vertical components to obtain the initial horizontal and vertical velocities.

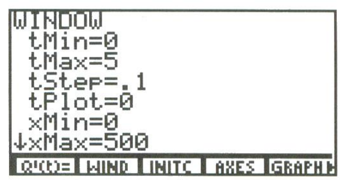


Figure 4.27

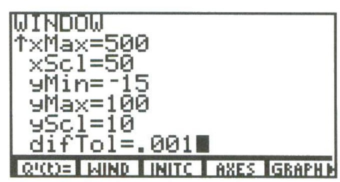


Figure 4.28

initial horizontal distance = $QI1 = 0$

initial horizontal velocity = $QI2 = 152\cos 20^\circ$

initial height = $QI3 = 3$

initial vertical velocity = $QI4 = 152\sin 20^\circ$

3. Select **Degree** mode in the mode screen and then enter the initial conditions (Figure 4.29).

   The trigonometric functions can be entered directly into the initial conditions editor. They are converted to decimal approximations when you press [ENTER]. (Figure 4.30)


Figure 4.29

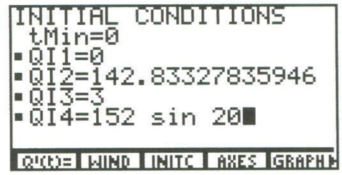


Figure 4.30

4. Since distance = **x** = Q1 and height = **y** = Q3, enter **x** = Q1 and **y** = Q3 in the axes editor. (Figure 4.31)

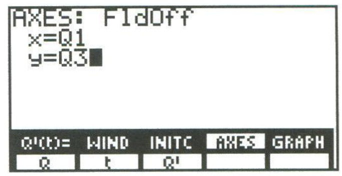


Figure 4.31

5. When you graph the solution, the display shows a ball and the path it follows. (Figures 4.32 and 4.33)

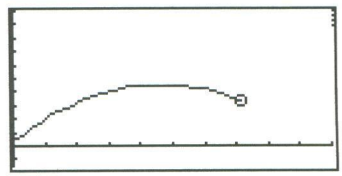

Figure 4.32

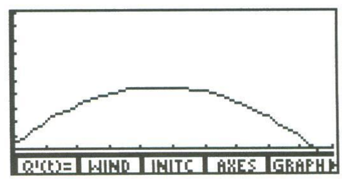


Figure 4.33

6. You can use the **TRACE** feature to see how far the ball travels. (Figure 4.34)

   The ball travels about 471 feet in about 3.3 seconds.

   In the next example, you will incorporate air resistance in the model for projectile motion.

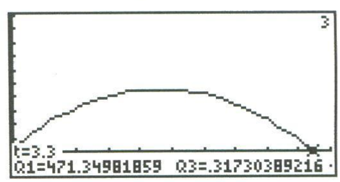


Figure 4.34

## Example 3: Modeling Baseball Trajectories with Air Resistance

How far will the baseball described in Example 2 travel with air resistance?

### *Solution*

Assume air resistance is a force that acts opposite of and directly proportional to the velocity of the ball. This is called *linear air resistance* and can be represented by the equation $f = -kv$, where $f$ is force, $v$ is velocity, and $k$ is a constant related to factors such as altitude, humidity and temperature.

Newton's second law gives us $f = ma$ where $f$ = force, $m$ = mass and $a$ = acceleration. When you set the force in Newton's second law equal to the force of air resistance and solve for acceleration due to air resistance you obtain

$$a = -\frac{k}{m}v\,.$$

Since the acceleration due to air resistance is opposed to the velocity it is sometimes called *deceleration.*

If

$$\frac{k}{m} = .05$$

for this baseball problem, then $a = {}^{-}.05v$ and our equations for horizontal and vertical acceleration from Example 2 will change to:

$$\text{horizontal acceleration} \quad = x'' = Q''1 = Q'2 = -.05Q2$$

$$\text{vertical acceleration} \quad = y'' = Q''3 = Q'4 = -32 - .05Q4$$

1. Make these changes in the differential equation editor. (Figure 4.35)


**Figure 4.35**

2. Use the same settings as Example 2 for the other menus and graph the solution to this new system of differential equations. Now when you trace, you can see that the ball doesn't travel as far. (Figure 4.36)

   With air resistance, the ball only travels about 422 feet. The hang time is about 3.2 seconds.

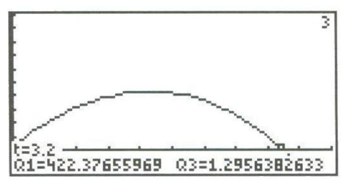


**Figure 4.36**

## Exercises

1. A ball was tossed straight up and its height was measured with a Texas Instruments CBL™. Here are the data:

| time (sec) | 0 | .04 | .08 | .12 | .16 | .20 | .24 | .28 | .32 | .36 | .40 | .44 | .48 | .52 | .56 | .60 | .64 | .68 | .72 | .76 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| height (meters) | .29 | .44 | .57 | .68 | .78 | .86 | .92 | .96 | .99 | 1.0 | 1.0 | 1.0 | .97 | .94 | .89 | .82 | .74 | .64 | .53 | .40 |

   a. Make a scatter plot of the data.

   b. Find the second degree polynomial that best fits the data. Graph this polynomial with the scatter plot.

   c. Graph the solution to the differential equation

$$\frac{d^2y}{dt^2} = -9.8$$

   that best matches the data.

   ***Hint:*** *The initial velocity is not zero. The middle coefficient of the second degree polynomial you found in part (b) gives you the initial velocity. This velocity will be entered in QI2.*

2. A baseball is hit when it is 3 feet above the ground. It leaves the bat with an initial velocity of 152 feet per second at an angle of 26° with the horizontal. How far will the ball travel if there is no air resistance?

3. How far will the baseball in Exercise 2 travel if the deceleration due to air resistance is $a = -.05v$ ?

4. How far will the baseball in Exercise 2 travel if the deceleration due to air resistance equals

$$a = -.05v^{\frac{5}{3}} ?$$

   This is nonlinear air resistance.