***NUMB3RS* Activity: Implicit Orbits**
Episode: "Hardball"

Topic: Implicit Differentiation **Grade Level:** 10 - 12
Objective: Use implicit differentiation to find the derivative of a circular orbit.
Time: 20 - 30 minutes

Introduction

In "Hardball," the FBI is searching for an amateur mathematician who has gone into hiding. Charlie hypothesizes that people who disappear and are still alive are like satellites that have lost their planet. Charlie explains that if the planet vanishes, the satellites would not have a focus to their orbit, and would travel in another direction, likely to another source of gravity.

One such satellite orbiting the Earth is the Geostationary Operational Environmental Satellite (GOES). The GOES-10, also called the GOES-EAST, satellite is a high altitude geosynchronous orbiting satellite used for weather observations. If the Earth vanished, this satellite would maintain the same velocity but would travel at a path tangent to its orbit. In this activity we will focus on the direction of that travel by calculating the slope of the tangential path.

Discuss with Students

The GOES-EAST satellite appears stationary in the sky because its nearly circular orbit is located in the Earth's equatorial plane and matches the rotation of the Earth. Because of this circular orbit, its path can be expressed as the equation of a circle with the center of the Earth as the center of the circle. Help students visualize this equatorial plane so that they may see how the problem can be represented as a two-dimensional problem of a three-dimensional model. This will also help students understand why the radius of the satellite is the sum of the radius of the Earth and the altitude of the satellite.

Because of the difficulty students may have with questions 3, 4, 6, and some of the Extensions, you may want to spend a little more time reviewing their solutions.

Problem #3

$$x^2 + y^2 = 1{,}777{,}802{,}896$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Problem #4

$$x^2 + y^2 = r^2$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Make sure students understand that while the slopes of the tangent lines of these orbits would be equal, the orbits themselves would not be the same, and the respective *x*'s and *y*'s would be dependent on the radii of the orbits.

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Problem #6

$$\frac{2x}{a^2}+\frac{2y}{b^2}\left[\frac{dy}{dx}\right]=0$$

$$\frac{2y}{b^2}\left[\frac{dy}{dx}\right]=-\frac{2x}{a^2}$$

$$\frac{dy}{dx}=-\frac{xb^2}{ya^2}$$

Extension #1

Find the linear velocity by dividing the circumference of the orbit by the time of one rotation. (Because the satellite is geosynchronous, it completes one orbit in 24 hours.) Then convert this velocity to meters/second.

$$\frac{(2\pi)42{,}164\text{ km}}{1\text{ day}}\bullet\frac{1\text{ day}}{24\text{ hour}}\bullet\frac{1\text{ hour}}{60\text{ min}}\bullet\frac{1\text{ min}}{60\text{ sec}}\bullet\frac{1000\text{ m}}{1\text{ km}}=3{,}066.2\text{ m/s}$$

Extension #2

$$V_e=\sqrt{\frac{2\left(6.673\times10^{-11}\,\frac{\text{m}^3}{\text{kg}\,\text{s}^2}\right)\left(6\times10^{24}\text{kg}\right)}{\left(42164\text{ km}\right)\left(\frac{1000\text{ m}}{1\text{ km}}\right)}}$$

$$V_e=4357.9\text{ m/s}$$

The lower bound of the velocity of the satellite orbits would be a circular orbit while the upper bound of the velocity would be less than the escape velocity. Velocities between these bounds would create varying elliptical orbits. This can be illustrated with the traditional representation of a conic section and how the eccentricity affects the shape of the ellipse. This also creates an interesting visual for the students to explore. To further explore this connection, visit the Web site below.
http://csep10.phys.utk.edu/astr161/lect/history/newtonkepler.html

Student Page Answers:

1. *6378 + 35786 = 42164 km* **2.** $x^2 + y^2$ = 1,777,802,896 **3.** $\frac{dy}{dx}=-\frac{x}{y}$ **4.** $\frac{dy}{dx}=-\frac{x}{y}$ **5.** *The derivative of the radius, regardless of its value, is zero.* **6.** $\frac{dy}{dx}=-\frac{xb^2}{ya^2}$

Extension Page Answers:

1. *3066.2 m/s* **2.** *4357.9 m/s* **3.** *1291.7 m/s* **4.** *It would fall back to Earth*. **5.** *It would be an elliptical orbit.*

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Name: ______________________________ Date: ______________

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The GOES-EAST appears to be stationary in the sky because its nearly circular orbit is located in the Earth's equatorial plane and matches the rotation of the Earth. Because of this circular orbit, its path can be expressed as the equation of a circle with the center of the Earth as the origin.

GOES-EAST Satellite

Radius of Earth	6,378 km
Altitude of GOES-EAST	35,786 km
Position	75 W Longitude and the Equator
Launched	April 25, 1997

[Source: **http://noaasis.noaa.gov/NOAASIS/ml/genlsatl.html**]

1. What is the radius of the circle the GOES-EAST traces?

2. If the center of the Earth is the origin (0, 0), what is the equation for the orbit?

If the Earth were to vanish, the path of the satellite would be a straight line tangent to the circular orbit of the satellite. To find the slope of this tangent line at any point on the orbit, calculate the derivative of the equation of the orbit. Rather than explicitly solving the equation for a variable before calculating the derivative, use the implicit form of the equation and find the derivative of the equation with respect to x (dy/dx).

3. Find the derivative of the equation for the GOES-EAST orbit.

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The National Oceanic and Atmospheric Administration (NOAA) operates polar orbiting weather satellites that travel in circular orbits around the Earth. One such satellite is NOAA-15.

4. Without knowing the altitude of the NOAA-15, calculate the derivative of the circular orbit.

5. Explain why the radius of the circular orbit does not affect the slope of the line tangent to the orbit.

Circular orbits are special cases of elliptical orbits. In fact, the paths of all of the satellites around the Earth are elliptical orbits. The general form of the equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

6. Find the derivative of this general case using implicit differentiation.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Introduction

1. Using the information from the activity, calculate the current linear velocity of the GOES-EAST satellite and express your answer in meters/second.

If the Earth does not vanish, then in order for the GOES-EAST satellite to leave its orbit it would have to have a great enough velocity to overcome the pull of gravity back to the Earth. This velocity is called the *escape velocity*, V_e. This is represented by the equation

$$V_e = \sqrt{\frac{2GM}{R}}$$

where G is the gravitational constant ($6.673 \times 10^{-11} \frac{\text{m}^3}{\text{kg s}^2}$), M is the mass of the Earth (6×10^{24} kg), and R is the radius of the satellite from the center of the Earth (in meters).

2. Calculate the escape velocity for the GOES-EAST satellite.

3. How much greater would the velocity of the GOES-EAST need to be to escape the Earth's orbit?

4. What would happen to the GOES-EAST satellite if the velocity were less than the velocity required to maintain a circular orbit?

5. Describe the orbit of the GOES-EAST satellite if the velocity was greater than the velocity required for a circular orbit, but less then the escape velocity.

Additional Resources

- The Web site for the National Oceanic and Atmospheric Administration (NOAA) is **http://www.noaa.gov**.

- Explore the Gravity Simulator at this Web site: **http://www.arachnoid.com/gravitation/small.html**.

- Heavens-Above has information about satellites as well as times to see the International Space Station with the naked eye: **http://www.heavens-above.com**.

- The Physics Classroom has a module written to demonstrate circular and planetary Motion at **http://www.physicsclassroom.com/Class/circles/circtoc.html**.

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