Calculus

with the

TI-89

Sample Activity: Exploration 7

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WHO HIT THE LONGEST HOME RUN IN MAJOR LEAGUE HISTORY?

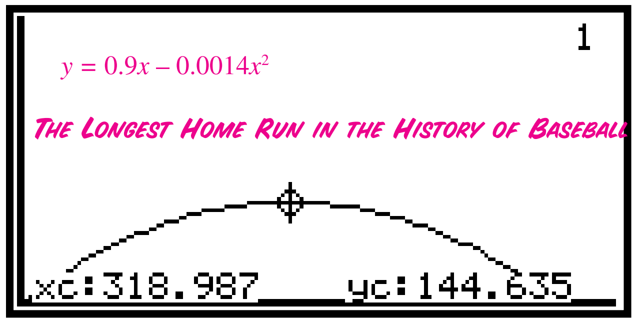

THE BETTMANN ARCHIVE

Mickey Mantle 1931-1996

On September 10, 1960, Mickey Mantle hit the longest home run ever recorded in regular-season major league baseball. In a game between the New York Yankees and the Detroit Tigers at Briggs Stadium in Detroit, he sent the ball into a parabolic orbit. The trajectory of the ball is given by the equation

$$y = 0.9x - 0.0014x^2$$

where x represents the horizontal distance (in feet) and y the vertical distance (in feet) of the ball from home plate. During his career with the New York Yankees (1951-68), Mantle hit a total of 18 home runs in World Series play. This is still a major league record.



WORKED EXAMPLE 1

a) Graph the trajectory of baseball's longest home run.
b) Determine the maximum height reached by the ball.
c) Determine how far the ball landed from home plate.
d) Determine the angle at which the ball left Mickey Mantle's bat.

SOLUTION

a) We define $y1(x) = 0.9x - 0.0014x^2$ and set the window variables to $0 \le x \le 700$; $0 \le y \le 405$. When we graph $y1(x)$, we obtain the display above.

b) To approximate the maximum height of the ball we can trace (F3) along the curve to the point (318.98…, 144.63…) shown in the display. This tells us that the maximum height is about 145 feet. For a better estimate of the maximum, we press **F5** **4** obtaining the prompt **Lower Bound?** We then trace backwards along the curve to any point left of the maximum and press **ENTER**. This yields the prompt **Upper Bound?**. We then trace to the right of the maximum and press **ENTER**. The display reveals that the maximum in the interval defined by the lower and upper bounds is 144.64….confirming that the maximum height is about 145 feet.

Alternatively, we can actually *calculate* the maximum height by observing that the height is a maximum when the slope of the trajectory is 0; that is when the derivative of $y1(x)$ is 0. We define $f(x) = 0.9x - 0.0014x^2$, so $f'(x) = 0.9 - 2(0.0014)x$ Solving $f'(x) = 0$, yields $x = 0.9/(0.0028)$ or 321.428…. Evaluating $y1(321.428)$ yields 144.64… as above.

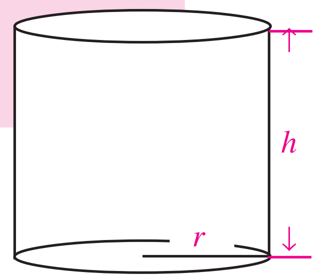
c) To approximate the length of the home run, we can trace along the graph to (641.17…, 1.50…) where the trajectory appears to intersect the x-axis. This yields a distance of 641 feet. To actually *calculate* the length, we can solve $y1(x) = 0$ using the formula for the roots of a quadratic equation, or by entering solve **(y1(x) = 0, x)** on the command line. In either case we obtain $x = 642.857\ldots$, indicating that the horizontal distance traveled by the ball was about 643 feet.

d) The slope of the trajectory at $x = 0$ is $f'(0)$, i.e. the tangent of angle at which the ball left Mickey Mantle's bat. But $f'(0) = 0.9$. The angle of inclination at impact was $\tan^{-1}(0.9) \approx 42°$.

The recent realization that the world's natural resources are in limited supply has made conservation an international priority of the 1990's. One of the world's most valued commodities which is now in short supply is tin.The worked example below is typical of many types of problems that applied mathematicians and engineers confront in the design of containers and packaging.

WORKED EXAMPLE 2

A cylindrical tin can must be designed to hold 300 *mL* of liquid. What dimensions for the can would require the minimum amount of tin? (Given the dimensions to the nearest hundredth of a centimeter.)



$$\underbrace{S = 2\pi r^2}_{\text{Total area of top and bottom}} + \underbrace{2\pi rh}_{\text{Area of the curved surface}}$$

SOLUTION

We assume the tin can is to be a right circular cylinder (for stacking purposes). We denote its height and the radius of its base in centimeters by h and r respectively.

We must find values of r and h which minimize the surface area S while maintaining a volume of 300 cm^3 (Recall $1\ cm^3 = 1\ mL$.) That is, we must minimize S subject to the condition $\pi r^2 h = 300$ where $S = 2\pi r^2 + 2\pi rh$.

Since $\pi r^2 h = 300$, we can substitute $\frac{300}{\pi r^2}$ for h in the formula for S to obtain: $S = 2\pi r^2 + \frac{600}{r}$
To graph S as a function of r on the TI-89, we represent r by the variable x and S by $y1(x)$.

We enter in the Y = Editor: $y1(x) = 2\pi x^2 + \frac{600}{x}$. To determine the appropriate window settings, we press: ◆ [TABLE] and scan the values of $y1(x)$ for $x = 1, 2, \dots 5$ as shown in the display. We see that $y1(x)$ decreases from $x = 1$ and begins to increase again near $x = 4$. We set the viewing window variables to $1 \le x \le 7$; $200 \le y \le 400$. After graphing $y1(x)$, we press: F5 3 to access **Minimum** on the calculus menu. We then respond to the prompts **Lower Bound?** and **Upper Bound?** as in *Worked Example* 1 and we obtain the display shown below. We observe that the minimum surface area is about 248 cm^2 and this occurs when $r \approx 3.63$ *cm* and $h \approx 7.25$ *cm*.(In general, the surface area is a minimum for $h = 2r$.)

x	y1
1.	606.28
2.	325.13
3.	256.55
4.	250.53
5.	277.08

x=5.

To *calculate* the minimum, we observe that the minimum occurs when the slope of the curve is 0. That is, when the derivative of $y1(x)$ is 0. We access **d(** on the calculus menu and enter the commands as shown to obtain this display.

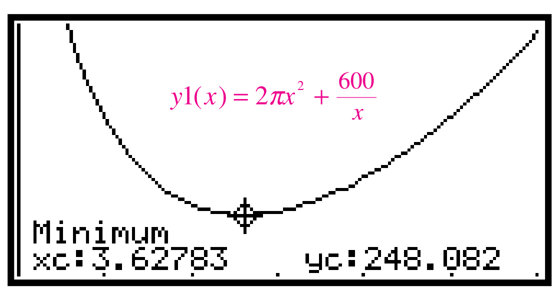
$$\blacksquare\ \frac{d}{dx}(y1(x)) \qquad 4\cdot\pi\cdot x - \frac{600}{x^2}$$

d(y1(x),x)

We then set this derivative equal to 0 i.e. $4\pi x - \frac{600}{x^2} = 0$.

Solving for x yields $x = \sqrt[3]{\frac{150}{\pi}} = 3.62783167\dots$

We see that the estimate was very close to the true value.



Exercises & Investigations

❶. In *Worked Example* 1 on page 34:
a) What is the name of the curve that describes the trajectory of the baseball?
b) What is the horizontal distance of the ball from home plate when it reaches the highest point in its trajectory?
c) What is the slope of the secant which joins the point on the trajectory at $x = 0$ to the highest point in the trajectory?
d) What is the slope of the tangent to the trajectory at its highest point? Give reasons for your answer.
e) The equation of the trajectory of Mickey Mantle's home run is a mathematical model of the path of the baseball. Do you think this model is accurate? Explain.

❷. A function has a *local maximum at* $x = a$ if and only if $f(x) \le f(a)$ for all values of x close to a. The value $f(a)$ is said to be a *local maximum*.
a) Define a *local minimum*.
b) Does the function defined by the equation $y = x^3 + 8$, have either a local maximum or a local minimum?

❸. Graph the function $f(x)$ where $f(x) = 2x^3 - 13x^2 - 7x + 1$.
a) Select the Maximum and Minimum commands on the F5 menu of the graph screen to determine all the local maxima and minima.
b) Solve the equation $f'(x) = 0$ to find the local maxima and minima and compare with your answer in part a).

❹. Determine all the points at which each of these functions has a local maximum or minimum.
a) $f(x) = 3x^2 - 7x + 5$
b) $f(x) = x^4 + 2x^3 - 3x^2 - 4x - 4$
c) $f(x) = x^4 - 6x^3 + 12x^2 - 8x - 5$

❺. A local maximum or minimum is called an *extremum*. What is the largest number of extrema which $f(x)$ can have if $f(x)$ is:
a) a linear function? b) a quadratic function?
c) a cubic function? d) a polynomial of degree n?

❻. Write an equation to define a function of x which has a maximum when $x = 0$, and a minimum when $x = 4$. Is the function you have defined the only one satisfying these conditions?

❼. Two sides of a triangular sign are to be each 5 *m* long. What length of the third side (to the nearest centimeter) would yield the maximum possible area?

❽. Determine the coordinates of the point on the line with equation $y = 3x + 2$, which is closest to the origin, (0, 0).

❾. What point on the parabola with equation $y = x^2 - 4x + 3$ is closest to the origin?

❿. a) Determine the distance between the lines defined by equations $y = 2.7x - 13$ and $y = 2.7x + 16$. (Give your answer to two decimal places.)

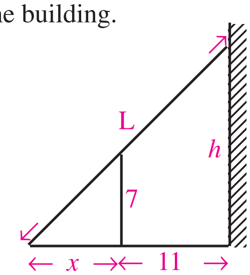
b) Check your answer to part a) by finding the point of intersection of the line defined by equation $y = 2.7x + 16$ and the line perpendicular to it and passing through the point (0, – 13).

⓫. In a heavy fog at 7:00 a.m., a cargo ship is 80 km due east of a luxury liner which is sailing due south at a speed of 42 *km/h*. If the cargo ship is sailing due west at 48 *km/h*, at what time will they attain their closest approach? How close will they be?

⓬. The total cost in dollars of producing x grummets is given by the formula $C = 0.35x^2 + 23x + 20$. Each grummet sells for $80.
a) How many grummets should be produced to achieve a maximum profit?
b) How much profit would this generate?

⓭. Ms. Chiu plans to put a rectangular swimming pool in her yard. Since one side of the pool will run along an existing fence it will be necessary for her to fence only three sides. What dimensions of the enclosed rectangle would maximize the contained area if Ms. Chiu has 35 *m* of fencing?

⓮. The diagram below shows a beam of length L laid across a wall 7 *m* tall to support the side of a building. The wall is 11 *m* from the building.

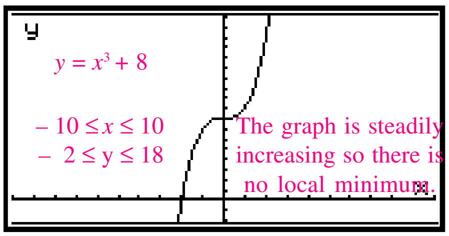


a) Denote the distance between the foot of the beam and the wall by x. Write an equation which relates x and h where h is the height of the upper end of the beam. (Use similar triangles.)
b) Write an equation for the length of the beam L in terms of x and h.
c) Use your equations in parts a) and b) of this exercise to express L in terms of the variable x only.
d) Graph L as a function of x and determine the minimum possible length of the beam.
e) How far is the bottom of the beam from the wall when the beam is the minimum possible length?
f) Does the function of x you graphed have more than one minimum? Intepret each minimum.
g) Is there a maximum length which the beam can have? Explain your answer.

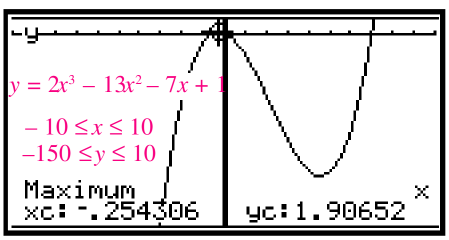
Solutions to Selected Exercises in Exploration 7

1\. a) a parabola
b) 321.43 feet
c) slope of the secant $144.64/321.43 \approx 0.45$
d) The slope of the tangent to the trajectory is 0 when the ball is at its highest point, because the vertical velocity of the ball is momentarily zero as it ceases to travel upward and begins to travel downward. That is, $\Delta y = 0$ over the interval Δx, so $dy/dx = 0$.
e) This is a model which would apply perfectly if the atmosphere were thin. Air resistance and lift generated by a spin on the ball can alter the shape of this trajectory, but the model is pretty accurate.

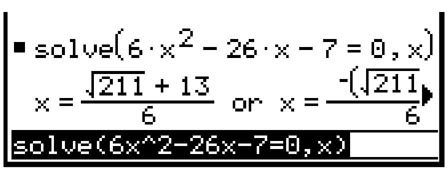
2\. a) The value $f(a)$ is said to be a *local minimum* of $f(x)$ if and only if $f(x) \geq f(a)$ for all values of x close to a.
b) The display shows that $y = x^3 + 8$ appears to flatten out near $x = 0$ but it has no local minimum because for $x < a, f(x) < f(a)$.



3\. a) The display shows that there is a local maximum of 1.9065… at $x = -0.254…$



By repeating the process, we find a local minimum of $-111.61…$ at $x = 4.587…$
b) The display shows there is a local maximum and a local minimum respectively at $x = (13 - \sqrt{211})/6$ and $x = (13 + \sqrt{211})/6$



When expressed in decimal form, the answers in a) & b) agree.

4\. a) minimum at (7/6, 11/12)
b) minima at (1, –8) and (–2, –8); maximum at (–1/2, –47/16)
c) minimum at (1/2, –107/16)

5\. a) None. The graph has constant slope and cannot change direction.
b) One. The graph is a parabola which opens upward or downward.
c) Two. The graph can cross the x-axis three times and change direction twice.
d) $n - 1$. Since $f'(x)$ is a polynomial of degree $n - 1$, the curve can change direction at most $n - 1$ times.

6\. $f'(x)$ is to be a quadratic function with zeros at $x = 0$ and $x = 4$, such that $f'(x)$ is changing from positive to negative near $x = 0$ and from negative to positive near $x = 4$. One such function is $f'(x) = x(x - 4)$; that is $f'(x) = x^2 - 4x$. One such function with this property is $f(x) = x^3 - 6x^2$. This function is not unique since multiplication by a constant (a vertical stretch) and/or addition of a constant (vertical translation does not change the extrema.

7\. Heron's formula states that the area A of a triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter of the triangle. Since A is a maximum when A^2 is a maximum, it is sufficient to maximize A^2. The sides of the triangle are 5, 5 and x *cm* long, so $s = x/2 + 5$. We must maximize the function $f(x) = (x + 10)(x)(x)(10–x)/16$. That is, $f(x) = x^2(100–x^2)/16$. This is a maximum when $x = 5\sqrt{2}$. The triangle has maximum area when the angle between the equal sides is $\pi/2$.

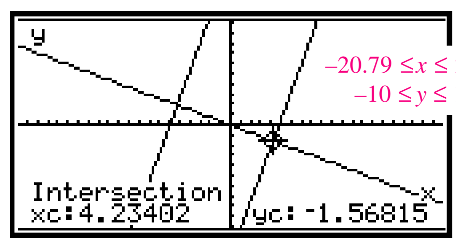
8\. We must minimize the distance from the point (x, y) to (0, 0), or what is the same, the square of this distance, i.e. $x^2 + y^2$ for all points that satisfy the condition $y = 3x + 2$. The function,

$$f(x) = x^2 + (3x + 2)^2 \text{ or } f(x) = 10x^2 + 12x + 4$$

has a minimum of 74/5 at $x = -3/5$. The closest point to the origin is (–3/5, 1/5).

9\. Proceeding as in exercise 8, we minimize the function $x^2 + y^2$, subject to the condition $y = x^2 - 4x + 3$. We find that the function $f(x) = x^4 - 8x^3 + 23x^2 - 24x + 9$ has a minimum at $x = 0.83…$ The y-coordinate of this point is $y = 0.38…$ The distance of this point from the origin is about 0.91…

10\. a) Clearly the two lines are parallel with slope 2.7. The line through the origin which is perpendicular to these two lines has equation $y = -(2.7)^{-1}x$. We graph these three lines and select 5 from the F5 menu to get the points of intersection as shown in the display.



The points of intersection are (–5.21, 1.93) and (4.23, –1.56). The distance between these two points is 10.07 (to 2 decimal places). You can find this distance directly on your TI-89 by pressing F5 followed by 9 on your graphing screen and then entering the x coordinates of the points of intersection on the line $y = -(2.7)^{-1}x$.

11\. The ships reach their closest proximity at 0.94 hours or 56.4 minutes after 7:00 a.m. At that time they are 52.68 km apart.

12\. The profit function to be maximized is revenue minus cost, i.e. $f(x) = -0.35x^2 + 57x - 20$. This function achieves its maximum when the daily production is about 81 grummets and this yields a profit of about \$2300.

13\. The area function is $y = x(35 - 2x)$. The area is maximized when the length is 17.5 meters and the width is 8.75 meters.