EXPLORATIONS

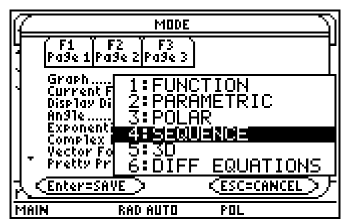
Chapter 9

Infinite Sequences and Series

An *infinite sequence* is a list or expression of the form: a_1, a_2, a_3, ... , a_n. An infinite sequence could be defined as a function with a domain consisting of the counting numbers. Since a sequence is a function, it can be represented symbolically, numerically, and graphically.

Before you begin

Because the domain of a sequence consists of only the counting numbers, a different graphing mode is used on the TI-89 called the Sequence graphing mode. Press [MODE] and set **Graph= SEQUENCE** before you complete the first two examples in this chapter.



Example 1: Investigating sequences numerically, graphically, symbolically

Investigate the behavior of the sequence

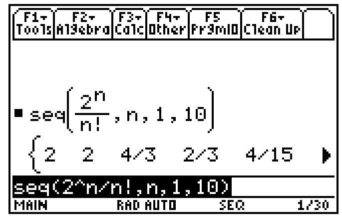
$$a_n = \frac{2^n}{n!}$$

for large values of n.

Solution

First, user the **seq** (sequence) command to look at the sequence numerically. Then graph the sequence to examine its behavior. Finally, use the limit(command to evaluate the sequence symbolically.

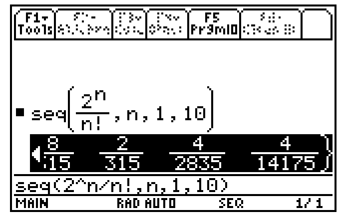
1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults.



2. Generate several terms of this sequence with the **seq(** command.

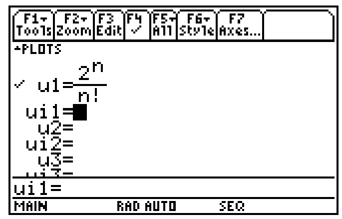
   [CATALOG] **seq(2** [^] **N** [÷] **N** [◆] [÷] [,] **N** [,] **1** [,] **10** [)] [ENTER]

3. You can see the rest of the sequence by pressing ⊙ to move up to the last line in the history window and then scrolling to the right by pressing ⊙.

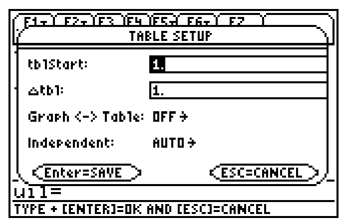


   It appears that the terms in the sequence are approaching 0 as n gets large.

4. Now use a table to represent the sequence. In the Y= Editor, enter the nth term expression for the sequence in $u1$. Clear $ui1$, if necessary.

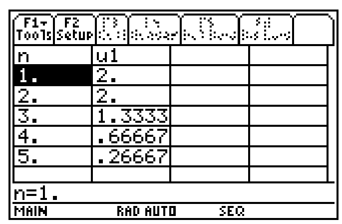


5. Press [◆] [TblSet] to display the TABLE SETUP dialog box, and then enter the values shown.

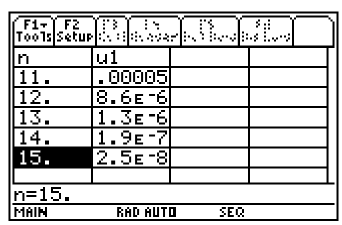


   The initial value of n is stored in **tblStart** and the increase in n from one row to the next in the table is stored in Δ**tbl**.

6. After entering these values, press [ENTER]. Then press [◆] [TABLE] to see the table.

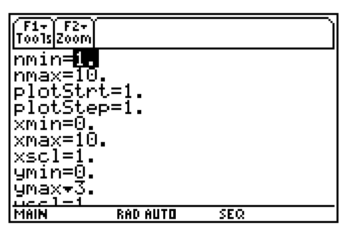


7. Scroll down the table a row at a time by pressing ⊙ or a page at a time by pressing [2nd] ⊙.



   By the 15th term, the sequence is very close to zero.

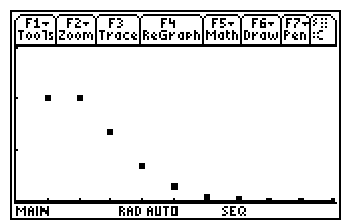
8. Now graph the sequence. Since the *nth* term is already stored in $u1$, you need only specify a viewing window before graphing the sequence. Press [◆] [WINDOW] and enter the values shown.



The window variables have the following functions:

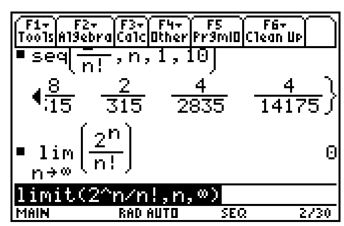
Variable	Function
nmin	The minimum or starting value of *n* used to generate the sequence
nmax	The maximum or final value of *n*
plotStrt	The value of *n* at which graphing begins
plotStep	The spacing between plotted points
xmin	The left side of the viewing window
xmax	The right side of the viewing window
xscl	The spacing between the tick marks on the *x*-axis
ymin	The bottom of the viewing window
ymax	The top of the viewing window
yscl	The spacing between the tick marks on the *y*-axis

9. Press [◆] [GRAPH]. The graph appears to converge to the x-axis. This is further evidence the sequence converges to zero.



10. Return to the Home screen to confirm this conjecture by evaluating the expression **limit(2^n/n!,n,∞)**.

    [CATALOG] **limit(2** [^] **N** [◆] [÷] [,] **N** [,] [◆] [∞] [)] [ENTER]



You now have numerical, graphical, and symbolic evidence that the sequence

$$a_n = \frac{2^n}{n!}$$ converges to zero as n gets large.

Example 2: Convergence of an infinite series

An *infinite series* is an expression of the form:

$a_1 + a_2 + a_3 + \ldots + a_k + \ldots$

Another way to represent an infinite series is with the notation:

$$\sum_{k=1}^{\infty} a_k$$

The partial sums of a series are helpful in understanding the behavior of the infinite series. These partial sums form a sequence.

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$\vdots$

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

If the sequence of partial sums converges to a limit S as n gets large, we say the infinite series converges to S. If the sequence of partial sums diverges, the infinite series diverges.

Determine if the infinite series

$$\sum_{k=1}^{\infty} \frac{2^k}{k!}$$

converges. If the series converges, estimate its sum.

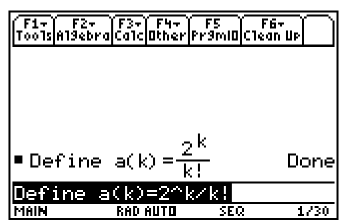
Solution

You can use the ratio test to establish the convergence of the infinite series. Then investigate the sequence of partial sums graphically and numerically to estimate the sum of the infinite series.

The ratio test says that if

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} < 1 \text{, the series } \sum_{k=1}^{\infty} a_k \text{ converges.}$$

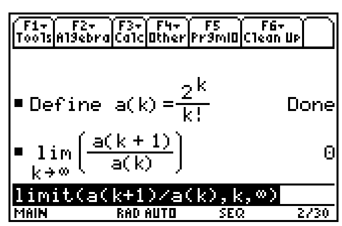
1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults.



2. Define the kth term in the series with the command **Define a(k)=2^ k/k!**.

   [CATALOG] **Define A** [(] **K** [)] [=] **2** [^] **K** [÷] **K** [◆] [÷] [ENTER]

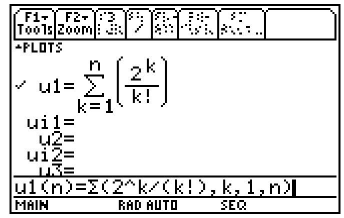
3. Apply the ratio test.



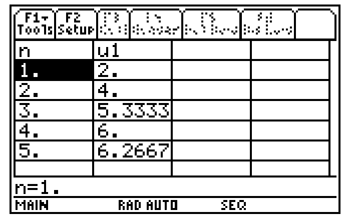
   [CATALOG] **limit(A** [(] **K** [+] **1** [)] [)] [÷] **A** [(] **K** [)] [,] **K** [,] [◆] [∞] [)] [ENTER]

   Since the limit is less than 1, the infinite series converges. Now estimate the sum of the infinite series.

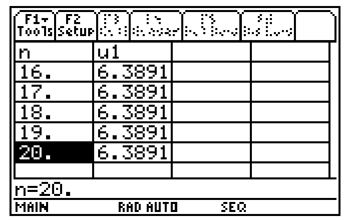
4. In the Y= Editor, enter the expression for the *nth* partial sum, **Σ(2^k/k!,k,1,n)**, in $u1$. To enter **Σ(**, press [2nd] [MATH] and then select **A:Calculus** followed by **4:Σ(sum**.



5. Set **tblStart =1** and **Δtbl=1.** In the TABLE SETUP dialog box, press [ENTER] and then press [◆] [TABLE] to display the table.

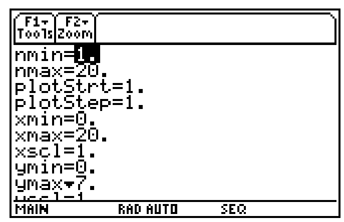


6. As you scroll down the table, you should see a possible limit to the sequence of partial sums.



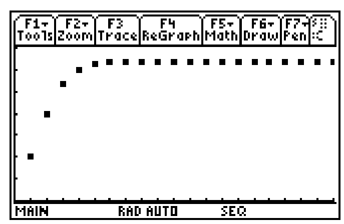
7. Graphing the sequence of partial sums helps to visualize the convergence of the sequence of partial sums.

   In the Window Editor, set up a [0,20] x [0,7] viewing window as shown.



8. Press [◆] [GRAPH] to see the graph.

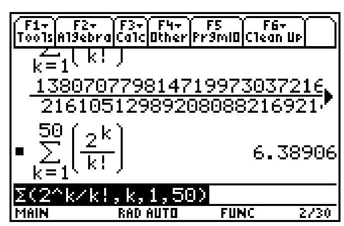
   The sequence of partial sums appears to level off after 20 terms.



9. Return to the Home screen to find the 50[th] partial sum by entering **Σ(2^k/k!,k,1,50)**.

   [CATALOG] **Σ(2** [^] **K** [÷] **K** [◆] [÷] [,] **K** [,] **1** [,] **50** [)] [ENTER]

   The result is exact but not very helpful. Press [◆] [ENTER] to repeat the command and obtain a decimal approximation.



Since the sequence of partial sums appears to converge to approximately 6.389, we estimate that

$$\sum_{k=1}^{\infty}\frac{2^k}{k!}=6.389.$$

Example 3: Taylor series for $f(x) = e^x$

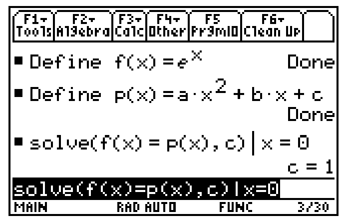
Taylor series give you polynomials that can be used to approximate other functions. This can be useful when the other function is difficult to evaluate or to manipulate symbolically. You use the fact that if two functions have identical first- and higher-order derivatives at a point, their graphs must be similar.

Determine values for the coefficients of $p(x) = ax^2 + bx + c$ so that the parabola is tangent to the curve $f(x) = e^x$ at $x = 0$.

Solution

The parabola will be tangent to the curve if both functions have the same value at $x = 0$ as well as the same first and second derivatives at $x = 0$.

1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults. In the MODE dialog box, set **Graph = FUNCTION**.

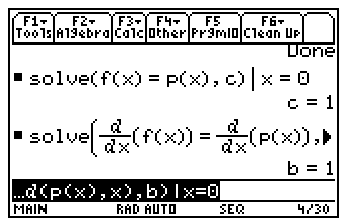


2. Enter the commands **Define f(x)=e^x** and **Define p(x)=a*x^2+b*x+c** to define the functions.

   [CATALOG] **Define F** [(] **X** [)] [=] [◆] [e^x] **X** [)] [ENTER]
   [CATALOG] **Define P** [(] **X** [)] [=] **A** [×] **X** [^] **2** [+] **B** [×] **X** [+] **C** [ENTER]

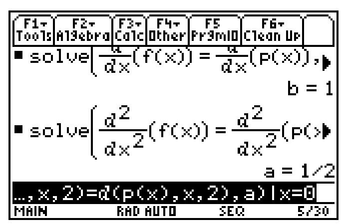
3. Enter **solve(f(x)=p(x), c)1x=0** to find the value of c.

   [CATALOG] **solve(F** [(] **X** [)] [=] **P** [(] **X** [)] [,] **C** [)] [|] **X** [=] **0** [ENTER]



4. Solve for the coefficient b that will make $f'(0) = p'(0)$ with the command **solve(*d*(f(x),x)= *d*(p(x),x),b)|x=0**.

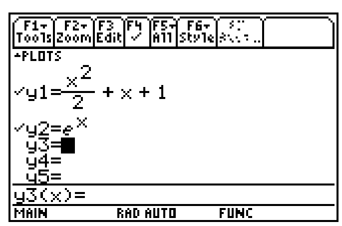
5. Solve for the coefficient a that will make $f''(0) = p''(0)$ with the command **solve(*d*(f(x),x,2)= *d*(p(x),x,2),a)|x=0**.



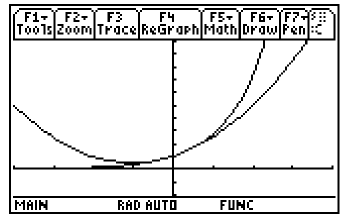
6. The coefficients are: $c = 1$, $b = 1$, $a = ½$. In the Y = Editor, enter the corresponding polynomial in $y1$. Enter e^(x) in $y2$.

7. Set up a [-4,4] x [-2,10] viewing window.

8. Press [◆] [GRAPH] to compare the graphs of the two functions.



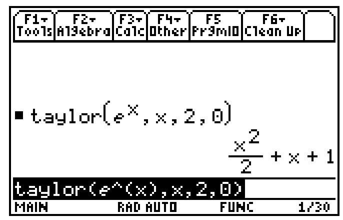
Notice that the polynominal is a good approximation of the exponential function for values of x near 0. This polynominal is called a *2^{nd} degree Taylor polynomial*.



You can generate this polynomial with the **taylor(** command from the **Calc** menu or the CATALOG. The parameters for this command are the function being approximated, the independent variable, the order of the polynomial, and the *x*-coordinate at which the polynomial is tangent to the function. If the last parameter is omitted, the TI-89 default value is 0.

9. Return to the Home screen and generate this polynomial with the **taylor(** command.

   [CATALOG] **taylor(** [◆] [e^x] **X** [)] [,] **X** [,] **2** [,] **0** [)] [ENTER]



Example 4: Integral of a Taylor polynomial

Find the 9^{th} degree Taylor polynomial for

$$f(x) = \tan^{-1} x$$

and compare with the integral of the 8^{th} degree Taylor polynomial for

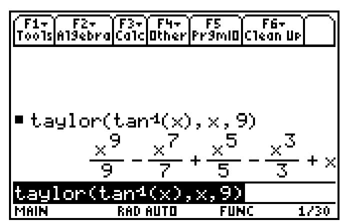
$$f(x) = \frac{1}{1+x^2}$$

Solution

1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults.

2. Enter the command **taylor(tan-1(x),x,9)**.

   [CATALOG] **taylor(** [◆] [TAN⁻¹] **X** [)] [,] **X** [,] **9** [ENTER]

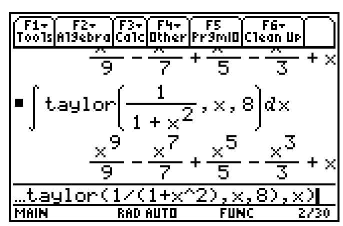


3. Now enter the command **∫(taylor(1/(1+x^2),x,8),x)**.

   [2nd] [∫] [CATALOG] **taylor(1** [÷] [(] **1** [+] **X** [^] **2** [)] [,] **X** [,] **8** [)] [,] **X** [)] [ENTER]

   Both commands produce the same result. This was expected since

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C.$$



Exercises

Determine whether the sequences in exercises 1 through 3 converge or diverge. If the sequence converges, estimate the limit to which it converges. Support your conclusion graphically, numerically, and symbolically.

1. $a_n = \dfrac{(-1)^{n-1}}{n}$

2. $a_n = \dfrac{n!}{n^n}$

3. $a_n = \dfrac{4n^2-1}{3n+2}$

Determine whether the series in exercises 4 through 8 converge or diverge. If a series converges, estimate the sum. Support your conclusion graphically, numerically, and symbolically.

4. $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{k}$

5. $\sum_{k=0}^{\infty} \dfrac{1}{k!}$

6. $\sum_{k=0}^{\infty} 4\dfrac{(-1)^k}{2k+1}$

7. $\sum_{k=1}^{\infty} \left(\dfrac{2}{3}\right)^k$

8. $\sum_{k=1}^{\infty} \left(\dfrac{4}{3}\right)^{k-1}$

9. Find the 8th degree Taylor polynomial for $f(x) = \dfrac{1+\cos 2x}{2}$ expanded about $x = 0$.

10. Find the 8th degree Taylor polynomial for $f(x) = \cos(2x)$.

    Add 1 to this result and then divide by 2. Use the **expand(** command to expand this result and compare it with the answer to Exercise 9.

11. Find the 5th degree Taylor polynomial for $f(x) = \ln(x)$ expanded about $x = 1$.

12. Find the 4th degree Taylor polynomial for $f(x) = \sin(x)$ expanded about $x = \dfrac{\pi}{2}$.