NUMB3RS Activity: Navigating Networks
Episode: "Blackout"

Topic: Networks, Directed Weighted Graphs   **Grade Level** 9 - 10
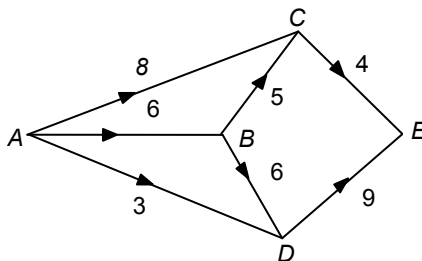**Objective:** Find maximum flow numbers in a directed weighted graph.
Time: 30 minutes

Introduction

In "Blackout," the team is investigating sabotage at power stations. These stations are part of a network that supplies electrical power to the city. The flow of things such as electricity, traffic, oil, and telephone calls from one place to another can be modeled using a network. The main challenge is to maximize this flow through the network and see how this maximum changes when one or more of the stations are eliminated.

Discuss with Students

This maximum flow problem is modeled by a **network** - a **directed, weighted graph**. A directed graph can be represented with vertices or nodes (points) and edges (arrows) connecting two vertices. Each edge has a direction, indicated by the arrow, and a **weight**. There is a **source** (a vertex with only outgoing flow) and a **sink** (a vertex with only incoming flow. In this graph, edge *AC* has weight 8; *A* is the source, and *E* is the sink.

Imagine that *A* is a power generating station; *E* is the station that supplies power to the local area; *B*, *C*, and *D* are intermediate substations; and the current flows through the edges (wires) *AC*, *AB*, *AD*, *BC*, *BD*, *CE*, and *DE*. The weights of the edges represent the current capacities (in thousands of amperes). The problem is to determine the maximum number of such units that can be sent from A to E. This quantity is the **maximum flow number** of this network.

To determine this maximum, the student will find a series of paths that carry the electricity from *A* to *E* in such a way that the sum of the number of units carried by these paths is a maximum. Path *A-C-E*, for example, can carry a maximum of 4 units since the capacity of *CE* is 4.
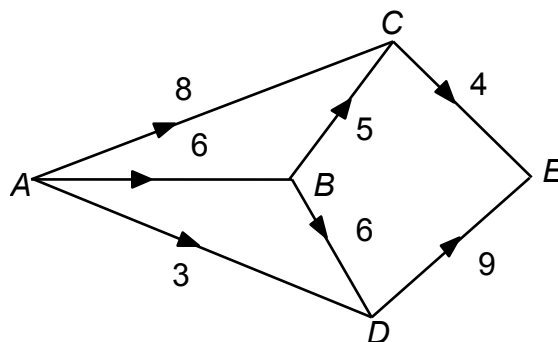
Student Page Answers:
*1.13; [ADE:3], [ACE:4], [ABDE:6] **2.** 12; A-B-D-E will change to 5 since the remaining capacity of DE is 5 after A-D-E is chosen. **3.** 13 Only 9 units can reach D, so the full capacity of DE cannot by used.*
4a. 4, since the capacity of DE is 4. 4b. A-B-D-E is now 2 since the remaining capacity of AB has been reduced to 2 (or the 6 units from A arriving at B split into 4 units to C and 2 units to D and then E) and A-D-E is 3 for a total of 9. By making this choice, current can flow from A to C in edge AC but cannot continue to sink, E. 5. 9, [ADE:3], [ABDE:6] 6. D: the maximum flow is 4 from A-C-E or A-B-C-E.
7. Any path must eventually contain one of the edges connected to the sink. 8. 17 9a. G or E 9b. 9
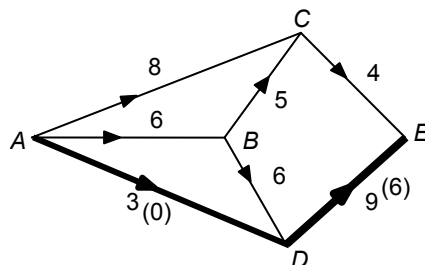
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NUMB3RS Activity: Navigating Networks

In "Blackout," the team is investigating sabotage at power stations. These stations are part of a network that supplies electrical power to the city. The flow of things such as electricity, traffic, oil, and telephone calls from one place to another can be modeled using a network. The main challenge is to maximize this flow through the network and see how this maximum changes when one or more of the stations are eliminated.
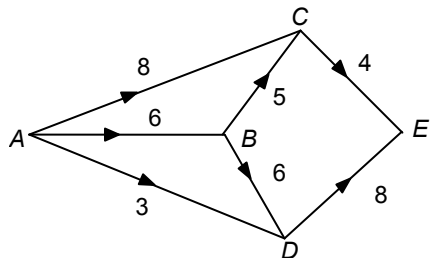

Consider the problem of finding the maximum flow number in the network above. Imagine that *A* is a power generating station (the **source** of the network); *E* is the station that supplies power to the local area (the **sink** of the network); *B*, *C*, and *D* are intermediate substations; and the current flows through the edges (wires) *AC*, *AB*, *AD*, *BC*, *BD*, *CE*, and *DE*. The weights of the edges represent the current capacities (in thousands of amperes). The problem is to determine the maximum number of such units that can be sent from *A* to *E*. This quantity is the **maximum flow number** of this network. To determine this maximum, find a series of paths that carry power from *A* to *E* in such a way that the sum of the number of units carried by these paths is a maximum.

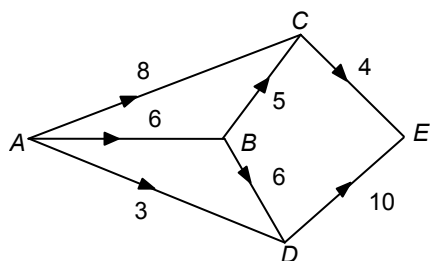**1.** Find the maximum flow number for this network.

First choose path *A-D-E* which can carry a maximum of 3 units (because the capacity of *AD* is 3). Denote this as [*ADE*: 3]. As 3 units pass through *A-D-E*, the remaining capacity of edge *AD* is now 0 and the remaining capacity of edge *DE* is now 6. Now find other paths and show that the maximum flow number for this network is 13.

2. Suppose the capacity of *DE* is changed to 8. What is the maximum flow number of the resulting network? Why?
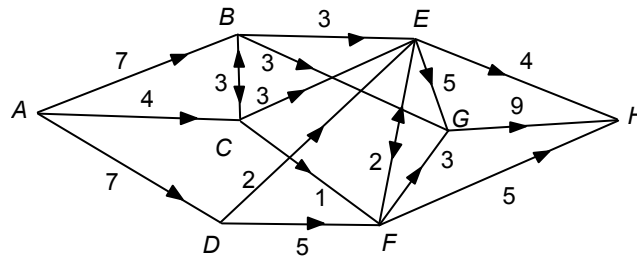

3. Suppose the capacity of *DE* is changed to 10. What is the maximum flow number of the resulting network? Why?

4. In Question 1, suppose you chose path *A-B-C-E* first.

    a. What is the maximum number of units this path can carry?

    b. Why will you be unable to find the maximum flow number of this network if you choose this path first?

5. Suppose station (vertex) *C* and all of the edges connecting to *C* are eliminated. What is the maximum flow number of the resulting network? Why?

6. To minimize the maximum flow number of the resulting network, which station should be eliminated? Why?

7. Explain why the following statement is true: "The maximum flow number cannot exceed the sum of the weights (capacities) of the edges connected to the sink."

8. Find the maximum flow number for the network below. Notice the flow can go in either direction along edges *BC* and *EF*.

Hint: It is possible to "split" the units arriving at a given vertex into parts so that some go along one edge and the rest go along another edge. In this case, start with the paths *A-D-F-H* which carries 5 units and *A-D-E-G-H* which carries 2. The 7 units arriving at *D* are split into 5 units going to *F* and 2 units going to *E*.

9. a. To minimize the maximum flow number in the resulting network, which station should be removed?

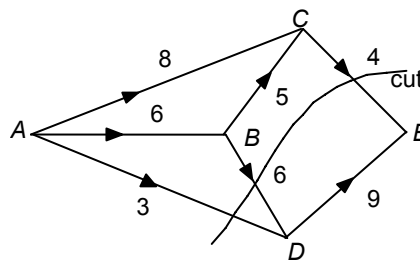**b.** What is the maximum flow number of the resulting network?

The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

<u>For the Student</u>

1. Suppose the capacity of each edge of a network with source *A* and sink *B* is 1. How can you characterize the maximum flow number of such a network in terms of paths from *A* to *B*?

2. There is a theorem relating a cut of a network to the maximum flow number. A **cut** of a network is a partition of the vertices into two subsets *U* and *V* so the source is in *U* and the sink is in *V*. The **capacity of a cut** is the sum of the capacities of all edges with one vertex in *U* and the other in *V*. A **minimum cut** is one whose capacity is as small as possible. The theorem states that: The maximum flow number equals the capacity of a minimum cut.
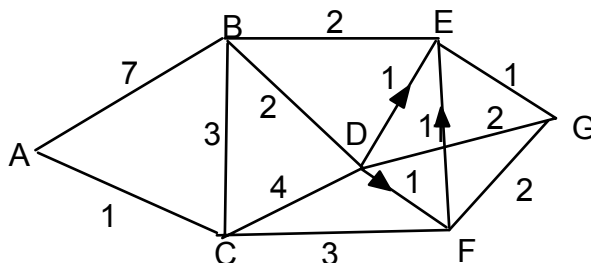
In this network, the cut shown has *U* = {*A, B, C*} and *V* = {*D, E*}. It is a minimum cut with capacity 3 + 6 + 4 = 13 which is also the maximum flow number.

   a. Find another minimum cut for this network.
   b. Find a minimum cut for the network used in Questions 8 and 9.

3. Choosing path *A-B-C-E* first in Question 4 did not ultimately lead to the paths for the maximum flow number. There is a "flow-augmentation algorithm" that iteratively constructs a series of paths that always lead to the maximum flow number. It is due to Ford and Fulkerson and more information can be found at the Web sites below:

   • **http://mathworld.wolfram.com/NetworkFlow.html**
   • **http://www-b2.is.tokushima-u.ac.jp/~ikeda/suuri/maxflow/Maxflow.shtml.en**

Related Problem

The **shortest path problem** is complementary to the Maximum Flow Problem. In this problem we seek the shortest path from *A* to *G* where the weights of the edges could be the time it takes to go from vertex to vertex as well as the distance or even the cost of a taxi. Notice that three of the edges are "one-way."

For more information, go to:
*   **http://mathworld.wolfram.com/ShortestPathProblem.html**
*   <u>For All Practical Purposes</u>, 7th ed. COMAP, Inc., 2006, W.H.Freeman, Chapters 1, 2