

## When is Tangent, tangent?

by – Paul W. Gosse

### Activity overview

*The unit circle (a circle radius 1 and centered at the origin) is a powerful tool for developing an understanding of trigonometry. This activity combines the ideas of unit circle, and a line tangent to the unit circle at  $0^\circ$  (in standard position), to explain how Tangent (the trig. ratio) is related to the concept of tangent to a figure (from geometry). The graph of  $y = \tan(x)$  for  $0^\circ \leq x < 90^\circ$  is also generated.*

*The intent is to briefly explore the mathematical history of the trigonometric ratio ‘tangent’ through an interactive data-gathering construction, simulating an experience that may well mirror how values of trigonometric functions may once have been approximated.*

### Concepts

*Slope, similar triangles, trigonometry, central angle, unit circle, solving angle of elevation problems.*

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### Teacher preparation

*Students should have studied the three primary trigonometric ratios in a right triangle setting and have an understanding of similar triangles.*

### Classroom management tips

*The activity is inter-active and exploratory in nature and well-suited for students working in pairs. The activity screens are set up for handheld view and may need re-positioning in computer view. Scaling view to 200% can assist when grabbing points in the activity.*

### TI-Nspire Applications

*Notes (with Q&A), Calculator, L&S, G&G.*

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### Step-by-step directions

*There are no precise step-by-step directions for this activity. The screens are self-explanatory. However, there are some comments given below.*

*The construction shown in 1.5 and used throughout the activity is limited in the sense that angle measures are  $0^\circ$  through  $90^\circ$ . While angles in quadrants one and four produce readable slopes in the activity, the nature of the original construction means angle values disappear in quadrants two and three. Students could be encouraged to determine those values given the existing construction.*

*Similarly, the graph of  $y = \tan(x)$  is restricted to  $0^\circ \leq x < 90^\circ$  in terms of the scatterplot on page 2.4.*

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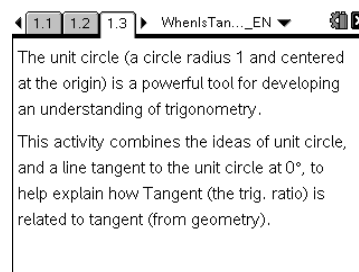
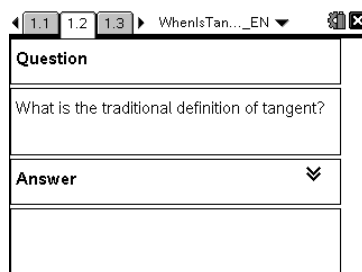
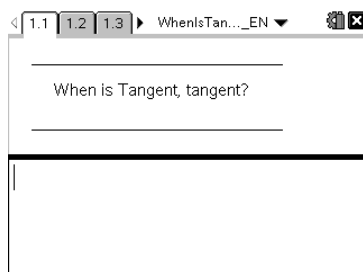
### Assessment and evaluation

- *Since the activity is historical in nature, assessment could consist of a journal activity inquiring about what was learned about the trigonometric ratio called tangent. Students might even be encouraged to construct a working tangent construction using paper or other materials as a class project.*
- *Answers: [1.2]  $\tan(x) = \frac{\text{opposite}}{\text{adjacent}}$ ; [1.4] Several extensions beyond the given measures are possible; [1.6] The slope of AH appears to be equal to TH and  $\tan(\text{HAT})$ ; [1.7] The slope of AC and AH are the same since then related sides involved in the slope are in the same ratio due to similar triangles; [ $\tan(\text{HAT})$  is TH over AT.  $\text{TH} = \tan(\text{HAT})$ . They are equal; [1.13]  $\tan(70^\circ) \approx 2.75$ ; [1.14] They are about the same. [1.15] 2.74748... Yes, they are about the same; [1.16] This would be the same as our previous calculation but scaled up by 10. So,  $10 \tan(70^\circ) \approx 27.5$  m; [1.20]  $50 \tan(70^\circ) \approx 137.4$  m; [1.21]  $50 \tan(50^\circ) \approx 59.6$  m.*

#### Activity extensions

- *Students should explore what values a construction like this one yields if the rotation brings the terminal arm into each of the other quadrants. In particular, the 'ruler' could be dropped to simulate the tangent of second quadrant angles.*
- *Encourage students to discuss what happens when C is rotated to exactly  $90^\circ$ . (At  $90^\circ$  extending the terminal arm of the angle results in a parallel to the tangent. Hence the arm never intersects the 'ruler', i.e., the tangent is  $90^\circ$  does not exist).*
- *Students who have studied the graphs of  $y = \sin(x)$  and  $y = \cos(x)$  know that their fundamental period is  $360^\circ$ . A construction such as the one in this activity could be used to develop reasoning for why the fundamental period of tangent is  $180^\circ$ , or to justify why the graph of  $y = \tan(x)$  has odd symmetry.*

#### Student TI-Nspire Document *WhenIsTangentTangent\_EN.tns*



1.3 1.4 1.5 WhensTan...\_EN

**Question**

Examine the construction on page 1.5 but resist dragging C for now. Think about what the elements of the construction are, and what measures are apparently available to us.

Then return to this page and read the notes provided as an **Answer** below.

1.3 1.4 1.5 WhensTan...\_EN

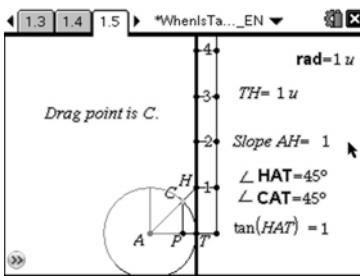
**Answer**

The construction on page 1.5 shows a unit circle with a radius extended to touch a vertical line **tangent** to the circle at T, which acts as a ruler graduated using the radius of the circle as its unit length. Triangle CAP lies inside the circle while triangle HAT involves the tangent line. Triangles CAP and HAT share the central angle CAP (HAT) and are similar. Also,  $\tan(\angle CAP$  or  $\angle HAT)$  is calculated and shown.

1.3 1.4 1.5 WhensTan...\_EN

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1.3 1.4 1.5 \*WhensTa...\_EN



Drag point is C.

rad=1 u  
TH=1 u  
Slope AH= 1  
 $\angle HAT=45^\circ$   
 $\angle CAT=45^\circ$   
 $\tan(HAT) = 1$

1.4 1.5 1.6 WhensTan...\_EN

**Question**

What other displayed measure(s) on the screen does the slope of AH appear to be equal to? Why?

**Answer**

1.5 1.6 1.7 WhensTan...\_EN

**Question**

How is the slope of AC related to the slope of AH?

**Answer**

1.6 1.7 1.8 WhensTan...\_EN

**Question**

What is the  $\tan \angle HAT$  using the  $\frac{opp}{adj}$  definition?  
Given the radius, AT, is 1, what is TH?  
So, how are TH and  $\tan \angle HAT$  related?

**Answer**

1.9 1.10 1.11 \*WhensTa...\_EN

Notice the tangent to the unit circle is drawn at  $0^\circ$  and is perpendicular to radius AT. TH is equal to the tangent of  $\angle HAT$  because  $\tan \angle HAT$  is  $\frac{TH}{AT}$  and  $AT=1$ . Hence,  $\tan \angle HAT = TH$ .

And, TH can be read directly from the graduated vertical tangent (the ruler).

1.8 1.9 1.10 WhensTan...\_EN

**So, on a unit circle, the tangent of a central  $\angle$  can actually be read from a line tangent to that circle by extending that terminal arm until it intersects the tangent**  
*(as long as the tangent is graduated using the radius of the circle as its unit length and is tangent to the circle at  $0^\circ$ ).*

1.9 1.10 1.11 WhensTan...\_EN

The ratio of vertical leg to horizontal leg in right triangle, similar to one we could construct here, would be equal to the value we would find using the tangent we constructed. The legs are respectively opposite and adjacent to the central angle, so the  $\frac{opposite}{adjacent}$  ratio is the same as the tangent value.

1.10 1.11 1.12 WhensTan...\_EN

Return to page 1.5 and drag point C around the first quadrant to locate other approximate values of tangent off the 'ruler'.

You can confirm your estimates using any scientific calculator.

1.11 1.12 1.13 WhensTan...\_EN

**Question**

Using the construction on page 1.5, estimate the tangent of  $70^\circ$ .

**Answer**

1.12 1.13 1.14 WhensTan...\_EN

**Question**

For a  $70^\circ$  central angle what would be the approximate length of TH?

How is this related to the last question?

**Answer**

1.13 1.14 1.15 WhensTan...\_EN

Using the calculator application below, determine  $\tan(70^\circ)$ . Was it close to your estimate from page 1.13?

0/99

1.14 1.15 1.16 \*WhensTa...\_EN

**Question**

What would the height be on a similar right triangle with a  $70^\circ$  central angle and  $AT=10$  m? Justify your answer.

**Answer**

1.14 1.15 1.16 \*WhensTa...\_EN

What would the height be on a similar right triangle with a  $70^\circ$  central angle and  $AT=10$  m? Justify your answer.

**Answer**

The triangle is scaled up 10 times  $\Delta HAT$  from the previous construction. So the height is 10 times  $\tan(70^\circ)$ .

1.15 1.16 1.17 \*WhensTa...\_EN

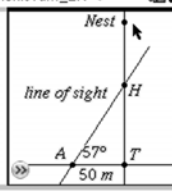
**Problem**

An starlings' nest is sighted from a spot 50 m from the base of a tree.

A model was drawn to avoid disturbing the starlings. Drag H on the model on the next page to find the angle of elevation to the nest.

1.16 1.17 1.18 \*WhensTa...\_EN

Use your earlier work to help approximate the tangent and use it to estimate the height of the nest (TH).



0/99

1.17 1.18 1.19 \*WhensTa...\_EN

**Question**

Insert a calculator page and use it to determine a more accurate height for the starlings' nest.

[Check your approach with the answer below.]

**Answer**

1.17 1.18 1.19 \*WhensTa...\_EN

[Check your approach with the answer below.]

**Answer**

Hint:  $\tan(70^\circ)$  is scaled up by a factor of 50 [using a similar triangles approach].

$50 \times \tan(70^\circ) \approx 137.374$ .

1.18 1.19 1.20 \*WhensTa...\_EN

**Question**

To the nearest tenth of a meter, how high is the nest?

**Answer**

0/99

1.19 1.20 1.21 \*WhensTa...\_EN

**Question**

If the same nest was at an angle of elevation of  $50^\circ$ , what would be the height of the nest?

**Answer**

1.19 1.20 1.21 \*WhensTa...\_EN

**Question**

If the same nest was at an angle of elevation of  $50^\circ$ , what would be the height of the nest?

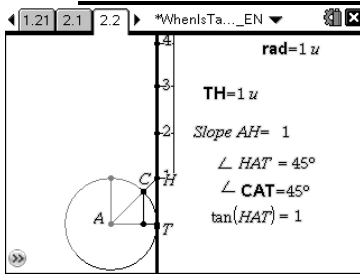
**Answer**

Approx. 59.6 m.

1.20 1.21 2.1 \*WhensTa...\_EN

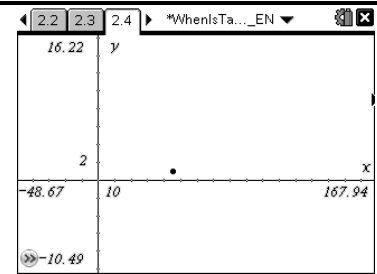
Drag C on the next page to capture values for tangent from  $0^\circ$  to  $90^\circ$ .

These tangent values will be captured in the spreadsheet that follows on page 2.3 and a scatterplot will be created on page 2.4 comparing the angle to the tangent value.



A	ang	B	ht	C
	= capture('cat,1)		= capture('th,1)	
1	45.		1	
2				
3				
4				
5				

A7 =45.



2.3 2.4 2.5 \*WhenIsTa...\_EN

You just developed a table of values (and a graph) for the tangent of angles of rotation on a unit circle in the first quadrant.

Tables like this were organized, copied by hand, and used for centuries to calculate distances mainly for use in navigation, engineering, and astronomy.

2.4 2.5 2.6 \*WhenIsTa...\_EN

Accuracy in measurement improved as tools became more precise during the industrial age, however, it wasn't until the 17th and 18th century that early calculating machines were used to create more accurate trigonometric and logarithm tables greatly improving accuracy in calculations in science and engineering.

2.5 2.6 2.7 \*WhenIsTa...\_EN

Some historians believe the earliest values in tangent tables were estimated from a circle made using a marked rope, with the rope then placed as the vertical tangent.

The Pythagoreans also had a marked rope :)