Moving Rectangular Objects around Corners

by Dave Slomer

Many calculus books ask something like, “Ignoring its width and height, what is the longest rod that can be moved around a 90° corner where two halls meet, if the hall is 5 feet wide on one side and 7 feet wide on the other side?” (See figure 1.) This is problem helps if you’re moving very, very long pieces of uncooked spaghetti and it gives good insight into moving long pieces of ½ inch diameter pipe. But what if you have to move a desk or a sofa? You certainly can’t ignore their widths. There is a more general solution that has more meaning to more real problems if you do <u>not</u> ignore the width. But first, the “spaghetti” problem…

Exercise 1: Find the longest piece of spaghetti (height and width are 0) that will fit around the corner described in the first sentence of this activity. Most calculus books offer the hint that the longest piece will be the shortest one that <u>remains in contact</u> with the inside corner and outside walls. That is, the spaghetti must stay in contact with points A, B, and C in figure 1. Similar triangles are the key (or the slope formula, for which coordinates have been assigned as an aid).

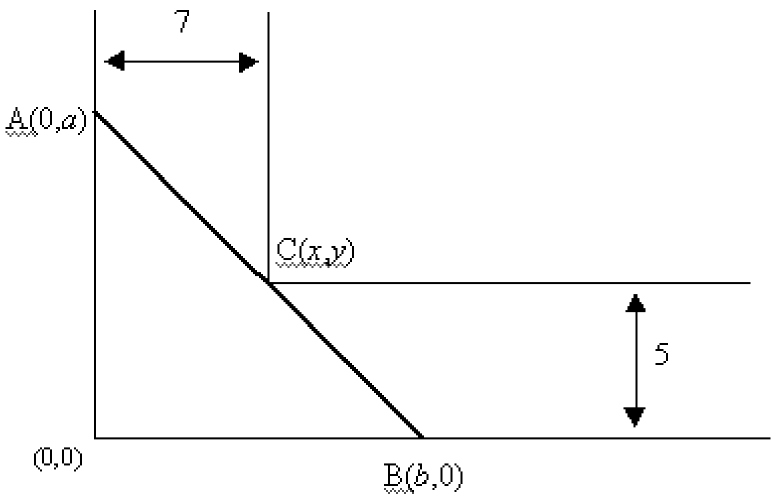
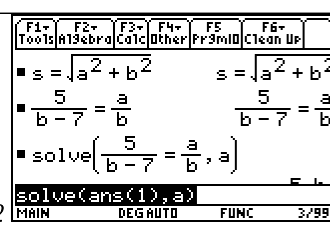


Fig. 1

- The spaghetti runs from A to B. What is a general formula for its length, **s**, in terms of *a* and *b*? Since **s** is a function of two variables, you must remove one of them. So…
- Draw a vertical (or horizontal) auxiliary line from C to an opposite wall. You now have similar triangles (or coordinates) that will give an auxiliary equation that will enable you to remove *a* or *b* from the length equation.

On the home screen of your TI-89, press [2nd][F6] to **Clear a-z**. Next, type the equation for **s** and the similar triangles (or slope) equation. Then solve the 2[nd] equation for **a**. Refer to figure 2.

Fig. 2



Next, replace the **a** in the formula for **s** by typing **ans(3)|ans(1)** as shown in the command line of figure 3. [You can, instead, cursor up and "grab" the **s** equation, press $\boxed{|}$, "grab" the **a** equation, and then press $\boxed{\text{ENTER}}$.]

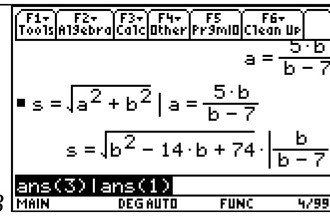


Fig. 3

The absolute value bars will wreak havoc on trying to take the derivative. To remove them, tell the '89 that **b>7**, by giving the command **ans(1)|b>7**, which will produce the screen in figure 4. (What is it about the problem that tells <u>you</u> that **b** has to be greater than 7?)

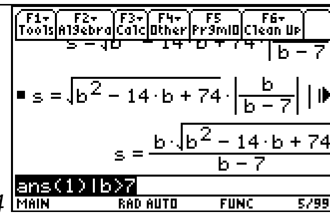


Fig. 4

Now that **s** is a function of <u>one</u> variable, you can find where the derivative is 0 and find the minimum point. To find the derivative, give the command in the command bar of figure 5. The '89 returns the <u>nasty</u> derivative on the right side along (and off the screen!) with the 0 you want it to equal on the left, along with a <u>warning</u>! If you give the command **solve(ans(1),b)** [shown in the command line of figure 6], you get a reasonable answer (see figure 6).

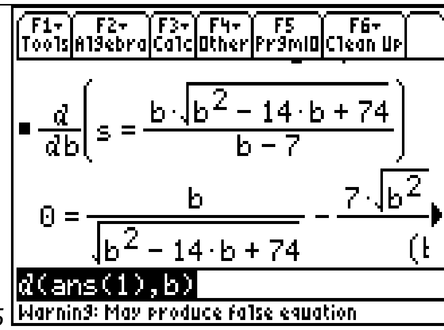


Fig. 5

But <u>what about that</u> <u>**warning**</u> about a possible **false equation**? Can you trust that <u>**symbolic**</u> answer? The <u>**numeric**</u> value that follows the warning <u>**seems**</u> reasonable, but … you <u>really need</u> a "second opinion". So…

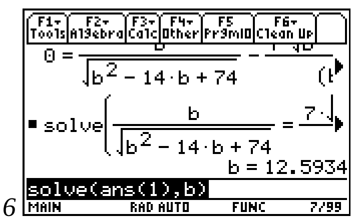


Fig. 6

<u>**Graph**</u> the length function by defining **y1** to equal **`x*√(x^2-14x+74)/(x-7)`**. Since 12.6 is the value for **b** (that is, **x**) of interest, set the window to include both that value and the likely approximate length. Your graph might resemble that in figure 7.

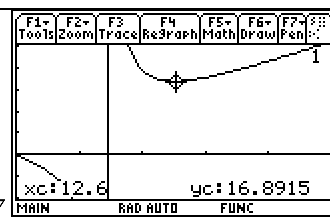


Fig. 7

The "second opinion" graph in figure 7 clearly shows that **y1** <u>is</u> a minimum if **x** (that is, **b**) is about 12.6, telling us that the length of spaghetti that would work shouldn't be much longer than 16.8 feet. (Since it's your '89 doing the work, you should check [symbolically] to see if **s''**(12.6) > 0.) (What is it about the graph tells you that **s''**(12.6) > 0?)

Now, on to the more general "corner" problem…

In figure 8, a *t*-unit wide, *L*-unit long rectangular object *OPWX* (maybe a desk) is to be moved around a corner with hall widths *a* and *b*. The movers will succeed if

$L \le |RU| + |PR| + |UW|$ is true. The <u>longest</u> object that will fit around the corner is the one for which $|RU| + |PR| + |UW|$ is a minimum.

Since the rectangle has width t, $OP = WX = ST = t$. In ΔRST, $\sin\theta = \dfrac{t}{RS}$, so that $RS = \dfrac{t}{\sin\theta}$. In ΔSTU, $\cos\theta = \dfrac{t}{SU}$, so that $SU = \dfrac{t}{\cos\theta}$.

Right ΔRSU has legs RS and SU, so that

$$RU = \sqrt{\left(\frac{t}{\sin\theta}\right)^2 + \left(\frac{t}{\cos\theta}\right)^2} = \frac{t}{\sin\theta\cos\theta}.$$

In ΔUVW, $\sin\theta = \dfrac{UV}{UW} = \dfrac{a - \frac{t}{\cos\theta}}{UW}$ so that

$$UW = \frac{a - \frac{t}{\cos\theta}}{\sin\theta}.$$

In ΔPQR, $\cos\theta = \dfrac{QR}{PR} = \dfrac{b - \frac{t}{\sin\theta}}{PR}$ so that

$$PR = \frac{b - \frac{t}{\sin\theta}}{\cos\theta}.$$

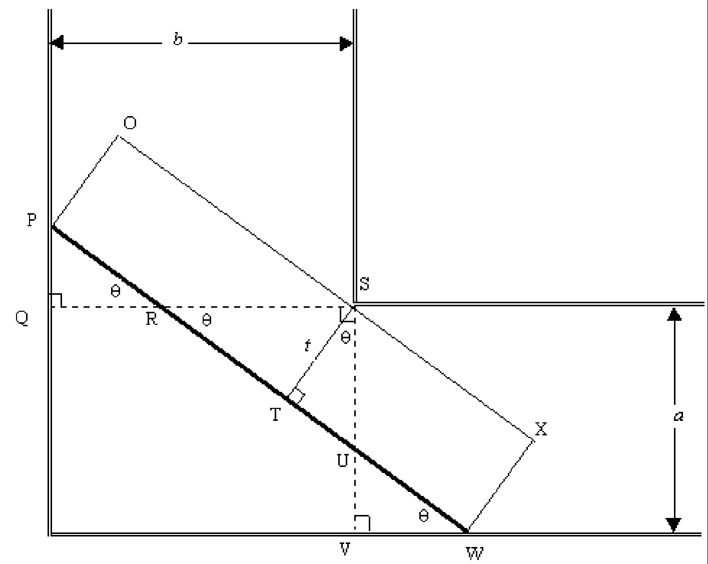


Fig.8

Adding the three segment lengths returns a none-too-complex fraction that is somehow nicer-looking if left in separate terms as in the first line of figure 9.

If $t = 0$, we solve the spaghetti problem. Might this trigonometric approach provide an easier way of doing Exercise 1? You'll get to see in Exercise 8.

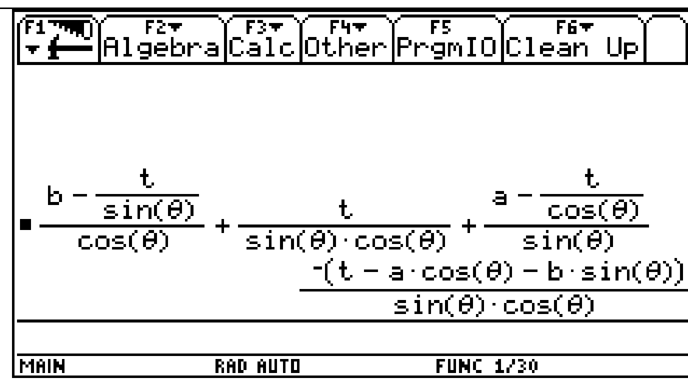


Fig.9.

The TI-89 cannot find a general formula for the minimum of this expression, but given suitable values for the hall and object widths, numerical solution is as easy as it was in the "spaghetti" problem. To easily supply the "desk" problem values, immediately after getting the expression in fig 9, give the command **ans(1)|t=2 and a=5 and b=7 →L**

(You might want to put your '89 into **Degree** MODE, just so the triangle angles are a little easier to interpret. Be sure to be in **Radian** MODE after finishing the activity.)

Exercise 2: The result of the command will be an expression in terms of θ only. Find the value of θ that gives minimum L, tell what that L is, and support your answer with both symbolic calculus results (derivatives and equation solving) and a graph. Interpret your final answer in a complete sentence or two, in terms of the problem's *parameters*—desk dimensions and hall widths.

Exercise 3: Most houses do <u>not</u> have 5- and 7-foot-wide halls. Some furniture is wider than 2 feet. Suppose your halls are 2.8 feet wide and that you have a 2.5-foot-wide table to move around a 90° corner. Will you make it? If so, with how much to spare? If not, by how much do you miss? Provide written symbolic and graphic support for your answer.

Exercise 4: You live in an unconventional house, whose halls do <u>not</u> meet at a 90° corner, meeting instead at 75° (the angle at the origin in figure 1 would be 75°, for example), but you do like spaghetti. The halls are 3 and 4 feet wide. Find the length of the longest piece of uncooked spaghetti that will pass through the halls. (Hint: Because there will be no right triangles, you will probably want to use the Law of Sines.)

Exercise 5: Now that you have done a <u>specific</u> problem, with measurements of 3 and 4 feet, <u>generalize</u>. Get a formula for the length to be minimized for halls *a* and *b* feet wide. Your '89 cannot provide a general formula for the minimum length. Why is that? Use your formula with the inputs of Exercise 4 and make sure that you get the same output that you did in that exercise. (Do you think you can write a program or create a text file for the '89 that would somewhat automate the process? Go for it.)

Exercise 6: In the unconventional house of Exercise 4, find the longest 2-foot-wide desk that will go around the corner.

Exercise 7: Generalize Exercise 6.

Exercise 8: There is some chance that you did not "Try it" as advised in the text just before figure 8. Try it.

Exercise 9: We have been ignoring the <u>height</u> of the object. In what ways might the height have an effect on the physical act of moving the furniture? In what ways might it affect the mathematics that has been done in this activity?

Exercise 10: Mathematics, when aided by technology, provides a way to somewhat easily arrive at solutions to many very complex problems. "Technology invites really hard mathematics problems." Discuss.

Calculus Generic Scope and Sequence Topics: Applications of Derivatives

NCTM Standards: Number and operations, Algebra, Geometry, Measurement, Problem solving, Connections, Communication, Representation