NUMB3RS Activity: A Recursion Excursion
Episode: "Protest"

Topic: Sequences defined recursively **Grade Level:** 10 - 12
Objective: Explore some interesting recursively defined sequences
Time: 30 minutes
Materials: TI-83/84 Plus calculator

Introduction

Written on one of Charlie's boards in a scene from "Protest" is the **Collatz Conjecture**, a problem Charlie is working on in his spare time. To form a Collatz sequence, begin by choosing a positive integer. If your integer is even, divide it by 2; if it is odd, multiply it by 3 and add 1. Repeat this process over and over. In all known cases, you will eventually reach the loop 1, 4, 2, 1, 4, 2. If your starting value is 6, for example, then the first 12 terms of your sequence are 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1.

More formally, the Collatz sequence is defined by $u_n = \frac{u_{n-1}}{2}$ if u_{n-1} is even, and $u_n = 3u_{n-1} + 1$ if u_{n-1} is odd, with a given value of u_1. The Collatz Conjecture, proposed by Lothar Collatz in 1937, is that this sequence reaches 1 (and this loop) for all starting numbers. In this activity we consider problems involving several number sequences, including the Collatz sequence.

Discuss with Students

A **number sequence** is an ordered list of numbers such as {2, 4, 6, 8, 10, …}. More formally, it is a function whose domain is a finite or infinite subset of the natural numbers. A **recursively defined sequence** is a sequence in which initial terms are given and then all terms after that are defined as a function of the preceding terms. The above sequence can be described using an explicit rule as $u_n = f(n) = 2n$ where n is a natural number or using a recursive rule as $u_n = u_{n-1} + 2$; $u_1 = 2$. Be sure to discuss with your students that you can find the 10th term of the sequence without calculating the previous 9 terms using the explicit rule, but need to calculate the 9th term to find the 10th term when using the recursive rule. If $u_n = f(n)$ for some function f with a given value of u_1, then as the value of n increases, the values of u_n will either converge to a finite limit L, diverge to ∞, or oscillate in some predictable pattern. Even though this activity is mainly concerned with sequences that eventually oscillate (like the Collatz sequence), you may wish to review the basic ideas with the following examples. All three types of sequences are further discussed in the Extensions.

Examples:

For each sequence:

- Find the first several terms of the sequence.
- As the value of n increases, do the values of u_n converge to a finite limit L, diverge to ∞, or oscillate in some predictable pattern?

1. $u_n = u_{n-1} + 2n - 1$; $u_1 = 1$ **2.** $u_n = (u_{n-1} + 9 / u_{n-1}) / 2$; $u_1 = 1$

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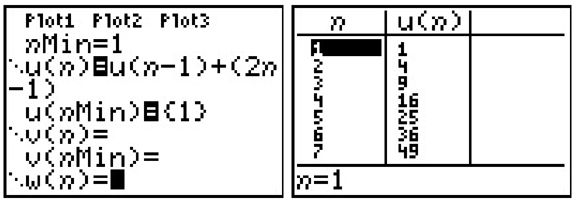
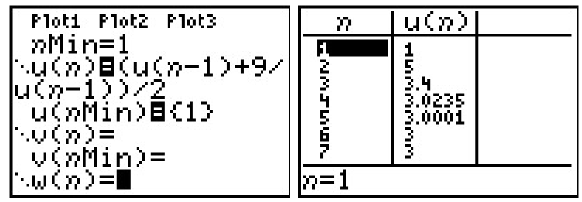
Solutions:

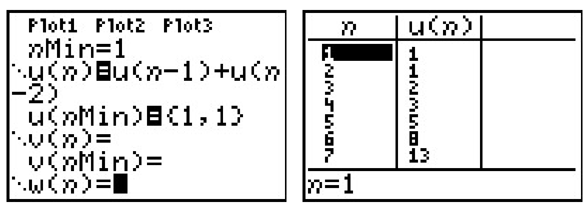
1. As shown below, this sequence is $u_n = n^2$, so it diverges to ∞.



2. As shown below, this sequence converges to $L = \sqrt{9} = 3$. This sequence is actually the Babylonian algorithm for finding square roots.



In these sequences, a given term depended only on the preceding term. In the Fibonacci sequence {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …} each term is the sum of the two preceding terms. Its defining equations are $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ with $F_1 = 1$ and $F_2 = 1$. You may wish to show your students how to find the terms of the Fibonacci sequence on a TI-83/84 Plus calculator as shown below (detailed instructions are given on Student Page 2). Note that `u(nMin)` is the list {1,1} = {u_2, u_1} which represents the values of F_2 and F_1 respectively.



Student Page Answers: 1a. *32, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4 ,2*
1b. *160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1*
1c. *7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1* **2.** *n* **3.** *n + 2m + 1*
4a. *{4, 5, 1.25, 0.25, 0.20, 0.80, 4, 5, 1.25, 0.25, 0.20, 0.80}* **4b.** *Answers vary, but in all cases the sequence is periodic with period 6.* **4c.** *The sequence is periodic with period 6: a, b,* $\frac{b}{a}, \frac{1}{a}, \frac{1}{b}, \frac{a}{b}, a, b, \ldots$ **5.** *The sequence is periodic with period 5:* $a, b, \frac{b+1}{a}, \frac{(a+b+1)}{ab}, \frac{(a+1)}{b}, a, b, \ldots$ **6a.** *1, 7, 13, 23, and 31 are happy numbers*
6b. *The sequence {s_n} eventually reaches the loop {4, 16, 37, 58, 89, 145, 42, 20}. So, if any term in the sequence {s_n} with s_1 = q. has the same digits as one of those 8 values, you know the number q is a sad number.* **6c.** *yes* **6d.** *no* **6e.** *no* **6f.** *no*

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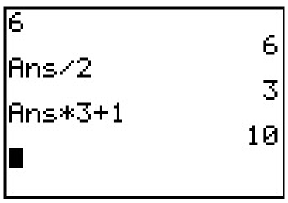
Name: ___________________________________ Date: ________________

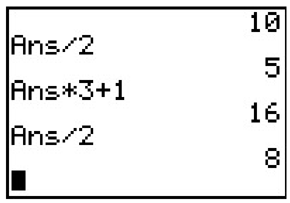
NUMB3RS Activity: A Recursion Excursion

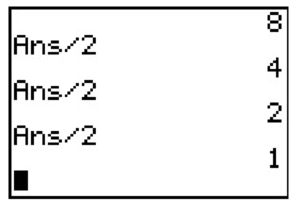
Written on one of Charlie's boards in a scene from "Protest" is the **Collatz Conjecture**, a problem Charlie is working on in his spare time. The problem is:

Choose a positive integer. If your integer is even, divide it by 2; if it is odd, multiply it by 3 and add 1. Repeat this process over and over. In all known cases, you will eventually reach the loop 1, 4, 2, 1, 4, 2.

For example, if your starting value is 6, then the first 12 terms of your sequence are 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, which you can calculate using the TI-83/84 Plus calculator as shown below.







Such a sequence is a **Collatz sequence**. A Collatz sequence is a type of a **recursively defined sequence** – a sequence in which initial terms are given and then all terms after that are defined as a function of the preceding terms.

1. Find the first 20 terms of the Collatz sequence beginning with each of the following values:

a. 32 **b.** 160 **c.** 7

The Collatz sequence can also be written as $u_n = \frac{u_{n-1}}{2}$ if u_{n-1} is even and $u_n = 3u_{n-1} + 1$ if u_{n-1} is odd, with $u_1 = p$. Most people feel that this sequence eventually contains a term equal to 1 for all values of p, but no one has ever been able to prove it. It is an **unsolved problem**. The Collatz sequence is sometimes called the "Hailstone Sequence" because the terms go up and down just like a hailstone in a cloud before crashing to Earth (reaching 1). The number of steps needed for the sequence to reach 1 is referred to as the **length** of the sequence. The length of the hailstone sequence with $u_1 = p = 6$ is 8.

2. In general, what is the length of the hailstone sequence with $p = 2^n$?

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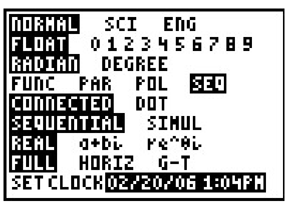
3. In question 1b, p was equal to 160. This value of p can be written as $p = 160 = \dfrac{2^5\left(4^2 - 1\right)}{3}$. In general, what do you think is the length of the hailstone sequence when $p = \dfrac{2^n\left(4^m - 1\right)}{3}$?

One of the most famous recursively defined sequences is the Fibonacci sequence {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...}. In this sequence, each term is the sum of the two preceding terms. The Fibonacci sequence can be defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, with $F_1 = 1$ and $F_2 = 1$.

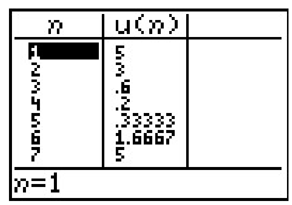
Suppose we consider a recursively defined sequence where a given term is the **quotient** of the two preceding terms. Specifically, let $u_n = \dfrac{u_{n-1}}{u_{n-2}}$ for $n \geq 3$, with $u_1 = a$ and $u_2 = b$.

You can use your calculator to help find the terms of a recursively defined sequence. The screenshots below show how to find the terms of the quotient sequence on a TI-84 Plus calculator when $a = 5$ and $b = 3$.



Press [MODE] and select SEQ.

Then press [Y=] to enter the sequence information. (Note: to enter the function name u, press [2nd] [7], since u is above [7].)



To display the terms of the sequence in a table, press [2nd] [TABLE].

4. a. Find the first 12 terms of $\{u_n\}$ when $a = 4$ and $b = 5$.

b. Experiment with more pairs of starting values a and b. Make a conjecture about the long-term behavior of this sequence.

c. Verify your conjecture using algebra. (Hint: The first 3 terms of the sequence are a, b, and $\dfrac{b}{a}$.)

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5. Consider the related sequence: $u_n = \dfrac{1+u_{n-1}}{u_{n-2}}$ for $n \geq 3$ with $u_1 = a$ and $u_2 = b$.

a. Experiment with different pairs of starting values a and b and make a conjecture about the long-term behavior of this sequence.

b. Verify your conjecture using algebra. (Hint: The first 3 terms of the sequence are a, b, and $\dfrac{(b+1)}{a}$.)

6. The number 19 is called a **happy number** because if you square each of its digits, find the sum of these squares, and then repeat, you eventually reach 1. We have:

$1^2 + 9^2 = 1 + 81 = 82$
$8^2 + 2^2 = 64 + 4 = 68$
$6^2 + 8^2 = 36 + 64 = 100$
$1^2 + 0^2 + 0^2 = 1 + 0 + 0 = 1$

More generally, let s_n denote the sum of the squares of the digits of s_{n-1} with $s_1 = q$. Values of q for which this sequence does not reach 1 are **unhappy** or **sad numbers**.

a. All but one of the numbers in the set {1, 7, 13, 23, 25, 31} are happy numbers. Find the happy numbers in this set.

b. For the number q in question 6a that is a sad number, what happens to the sequence $\{s_n\}$ with $s_1 = q$? How can this information be used to help you determine whether a number is happy or sad?

c. If H is a happy number, is 1,000,000H also a happy number?

d. Is the sum of two happy numbers always a happy number?

e. Is the product of two happy numbers always a happy number?

f. Is 2006 a happy year?

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The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

1. Consider the quotients of Fibonacci numbers: $u_n = \frac{F_n}{F_{n-1}}$ or $\{u_n\} = \left\{1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, ...\right\}$.

a. Express this sequence in recursive form.
b. This sequence converges to a famous irrational number. What is the number and why is it famous?

2. Install the program `HAILSTNE` on your TI-83/84 Plus calculator. To download this program, go to **http://education.ti.com/exchange** and search for “6619.”

Once the program has been loaded, run the program by entering a value for the first term, *p*. See if you can find any patterns between the length of the resulting hailstone sequences and the first term *p*.

3. Suppose we change the Collatz sequence to $u_n = \frac{u_{n-1}}{2}$ if u_{n-1} is even or $5u_{n-1} + 1$ if u_{n-1} is odd, $u_1 = p$. Only some values of *p* begin a sequence ending in 1.

a. Give three odd values of *p* whose sequence reaches 1.
b. What can happen for other values of *p*? (Hint: There are at least two possibilities.)

Additional Resources

- A comprehensive source of information on the Collatz Problem can be found at **http://www-personal.ksu.edu/~kconrow/**
- For an online version of a program similar to the calculator program Hailstone, see **http://did.mat.uni-bayreuth.de/personen/wassermann/fun/3np1_e.html**
- For more information on happy numbers, see **http://mathworld.wolfram.com/HappyNumber.html**
- A repository of “all” interesting sequences of whole numbers can be found at The On-Line Encyclopedia of Integer Sequences: **http://www.research.att.com/~njas/sequences/index.html**

Related Topic: Unsolved Problems

The Collatz Problem is one of many unsolved problems that exist today. Sometimes the most innocent question becomes a celebrated unsolved problem. One such problem is the Goldbach Conjecture: every even number can be written as the sum of two primes.

For more information on the Collatz Problem, the Goldbach Conjecture, and other currently unsolved problems, see Unsolved Problems: **http://mathworld.wolfram.com/UnsolvedProblems.html**

The most famous “unsolved problem” that has been solved in the past 25 years is Fermat’s Last Theorem: the equation $x^n + y^n = z^n$ has no positive integer solutions unless $n = 2$. For an interesting discussion of the path to the solution of this problem, see Fermat’s Last Theorem: **http://mathworld.wolfram.com/FermatsLastTheorem.html**

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