11. Fourier analysis with TI InterActive!

introduction

If you are familiar with Fourier analysis, you might know that (most) functions can be expanded into a Fourier series. This is a sum of sines and cosines that approximate a given function on a given interval. In this example we use *TI InterActive!* to (interactively) explore the connection between a function and its Fourier series. I must stress that this is just a light introduction, as the inner workings of Fourier analysis can be of quite a subtle nature (see note at the bottom).

example

We first define the function we will use. For example, let $f(x) := x$. This function can be altered at a later stage. We will use the interval $[-\pi, \pi]$ throughout. You may try to use other intervals in one of the exercises.

The so-called Fourier coefficients are defined by, respectively,

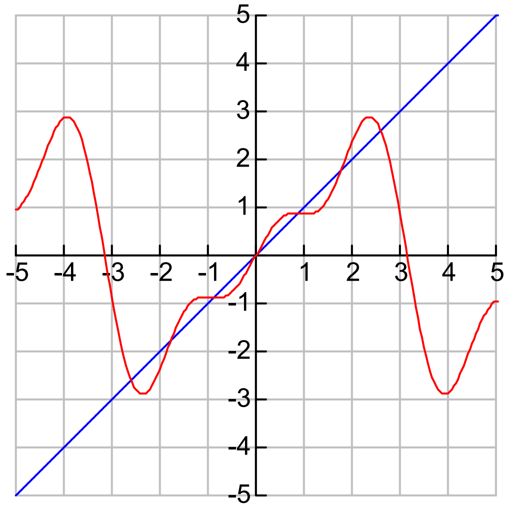
$$a(n) := \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} (f(x) \cdot \cos(n \cdot x))\, dx \quad \text{and} \quad b(n) := \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} (f(x) \cdot \sin(n \cdot x))\, dx .$$

Any text on differential equations or Fourier analysis will provide a definiton of these coeffecients. There are minor differences between textbooks, often through the use of different intervals or perhaps complex notation.

We can see that both of the Fourier coefficients depend on the function *f*. In Fourier theory, the intention is to add the coeffecients in an attempt to reconstruct the function *f*. Before we write the partial sum of the Fourier series, we need to agree on the degree of the sum. The higher the degree, the more accurately the partial sum will approximate the given function. We use a variable for this, so let $p := 3$. This can, of course, also be altered later on.

The partial sum *s(x)* in *f*'s Fourier series is given by the expression:

$$s(x) := \frac{1}{2} \cdot a(0) + \sum_{i=1}^{p} (a(i) \cdot \cos(i \cdot x) + b(i) \cdot \sin(i \cdot x)) .$$



This is where the variable *p* comes into play.

One important point, in terms of the theory is: does the sum converge to *f* if we use an infinite number of terms?

We can investigate this question through the use of TI InterActive!. We can plot the given function *f* with the corresponding Fourier series - partial sum in the same coordinate system.

The blue curve is the graph of the function *f* and the red curve is the graph of the partial sum.

This document will automatically be updated when you alter the function *f* or the degree of the partial sum.

Calculating Fourier series might be rather taxing on your computer's processor, so don't choose too high a value for the degree *p*.

As we see, the partial sum of *f*'s Fourier series seems to closely resemble *f* on the given interval. You can also use other intervals as Fourier series is not restricted to $[-\pi, \pi]$.

This interval is normally chosen to ease calculations.

Exercise 1: Change the function *f* and the degree *p* to see how different functions correspond to different Fourier series.

Exercise 2: Consult a text on calculus and find expressions for a general Fourier series on the interval from -*L* to *L*. Make a TI InterActive! document to illustrate the situation.

Note: Fully understanding Fourier series is quite a difficult task. One can use a functional analysis perspective, vector analysis or plain trigonometry. Rather difficult theorems on convergence and divergence may be required, as well as measure theory and Lebesgue integration. Fourier theory is actually responsible for changing the way one looks at subjects as convergence, function notion and the meaning of integration. Fourier's series made all these notions change some two hundred years ago and it took the leading mathematicians fifty years to settle matters. These theorie are beyond the level at which one is usually introduced to this kind of series. As a rule of thumb the functions that we use in practice are well suited for Fourier analysis. Functions that won't work are usually pathologic examples constructed to be counter-examples. Annoying mathematicians can find examples of Fourier series that do not converge and also those that converge to something other than the expected function. It is also possible to find trigonometric series that are not Fourier series for any function at all.