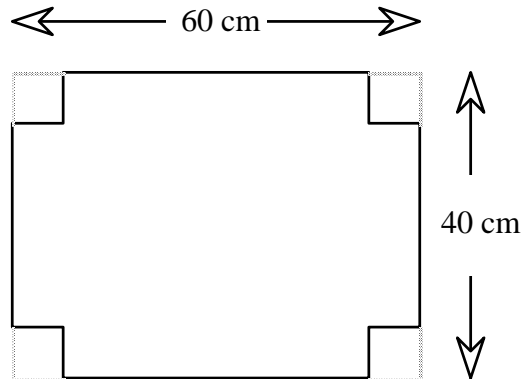Name _____

Date _____

Activity 9

Box It Up

People who design containers and boxes often try to create patterns that minimize the cost of materials or maximize the holding capacity of the final product. In this activity, you will decide on a design pattern that can be used to make a box with the largest volume.

The Problem

Ms. Hawkins, the physical sciences teacher at Hinthe Middle School, needs several open-topped boxes for storing laboratory materials. She has given the industrial technologies class several pieces of metal sheeting to make the boxes. Each of the metal pieces is a rectangle measuring 40 cm by 60 cm. The class plans to make the boxes by cutting equal-sized squares from each corner of a metal sheet, bending up the sides, and welding the edges.

This "classical" investigation takes a numerical or tabular look at finding the maximum volume of an open box constructed by folding a rectangular sheet of material with cut-out square corners. Activity 10 examines the same problem graphically. The combined lessons may take three or more 50-minute class sessions.

To better understand this problem, start by making a paper model of the kind of box Ms. Hawkins has requested. Using grid paper, mark off equal-sized squares at each of the corners and then cut out the squares. Fold the side tabs up and tape each edge together to form an open-topped box. Compare the general size and shape of your box with the boxes made by others in your classroom. Note that the size of the square cut from each corner of the paper determines the dimensions of the final product. What happens to the height, length, and width of the box as the size of the cut squares gets larger?

If you focus again on the boxes Ms. Hawkins needs, you can see that the industrial technologies class could make several differently shaped boxes.

✐ Go to the **Questions** section and answer #1 through #5.

Generating a Table of Volumes

Let's quickly summarize what you know at this point.

Ms. Hawkins wants the box with the largest possible volume. The size of the squares to be cut from each corner can be any whole number value from 1 through 19 cm in length.

If you enter the information you found earlier in a table, it may help you find a relationship between the dimensions of the box. Notice that the size of the cut-out square determines the height of the box. (Complete all but the last line of the table.)

Table 9.1. Table of Volumes

H	L	W	V
5	50	30	7500
10	40	20	8000
H	60–2H	40–2H	(60-2H)(40-2H)H

Height (cm) H	Length (cm) L	Width (cm) W	Volume (cm³) V
5	50		
10		20	
H			

Some values have been provided in the table as a check for your earlier solutions. Since you can choose the height of the box, H, height is called the *independent* variable. The other dimensions depend upon your choice of H, so they are *dependent* variables. When you pick H, the resulting length, L, is *60–2H*, for example, 60 cm – 2(5 cm) = 50 cm. The width, W, is found by using the expression *40 – 2H*. Finally, the volume, V, is the product of all three dimensions, or L∗W∗H.

Substitute in the other expressions for L and W in the product for finding volume:

V = L∗W∗H = (_____)(_____)H.   *V = (60–2H)(40–2H)H*

Complete the last row of the table using the expressions you have found for L, W, and V in terms of H.

✎ Go to the **Questions** section and answer #6.

You could add additional rows to the table by hand, but you have already found some important mathematical information about how the value of one variable can be used to determine the volume of the box. Using that information, you can use your graphing calculator to generate a table of the volumes of each of the nineteen possible boxes. Then the box with the largest volume can be found by searching through these values.

Using the Calculator

In order to generate a table of values, you must provide the calculator with three things:

- *an expression* written using only one variable, x

- *the value of x* to begin the table calculations

- *the size of the step* to be used in moving from one value of x to the next

It is important for the students to realize that they are still doing mathematics even though they aren't crunching out the values themselves. They found the mathematical relationship between height and volume that allows them the create the table using technology. They will still need to reason as they investigate the table of computed volumes.

For this situation, the expression we have written is *(60–2H)(40–2H)H*. We need only refer to the variable height as *x* instead of *H*. So *V=(60–2x)(40–2x)x*.

Earlier we decided that the shortest height of the box would be 1 cm, meaning that the smallest cut-out square would have a side length of 1 cm. This means that the table should start at *x=1*. The value of *x* should increase by 1 each time the next table entry is calculated. Then the table will contain whole number values for the cuts.

The instructions that follow will generate the desired table of volumes. After carrying out each set of instructions, your calculator window should look something like the window views shown at the right of the instructions.

1. Press $\boxed{\text{Y=}}$ to access the function editor menu.

2. If any expressions are listed, use $\boxed{\blacktriangledown}$ to move to each line containing an entry and press the $\boxed{\text{CLEAR}}$ key to delete the expression. Make certain the **Y1=** line is clear and the blinking cursor is at the right of the equal sign on this first line of the screen.

3. Press $\boxed{(}$ 60 $\boxed{-}$ 2 $\boxed{\text{X,T,}\Theta}$ $\boxed{)}$ $\boxed{(}$ 40 $\boxed{-}$ 2 $\boxed{\text{X,T,}\Theta}$ $\boxed{)}$ $\boxed{\text{X,T,}\Theta}$ to enter the expression for the volume of the box.

```
Y₁∎(60-2X)(40-2X
)X
Y₂=
Y₃=
Y₄=
Y₅=
Y₆=
Y₇=
```
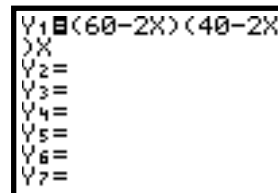
4. Press $\boxed{\text{2nd}}$ [TblSet] to access the Table Setup window.

5. Press $\boxed{1}$ to set **TblMin** to **1**.

6. Press $\boxed{\blacktriangledown}$ and press $\boxed{1}$ to set Δ**Tbl** to **1**.

```
TABLE SETUP
 TblMin=1
 ▵Tbl=1
Indpnt: Auto  Ask
Depend: Auto  Ask
```

The Greek letter delta, Δ, is typically used in mathematics to represent the idea of change.

7. If **Auto** is not highlighted on each of the last two lines, move down to the word **Auto** and press $\boxed{\text{ENTER}}$ for each line.

```
TABLE SETUP
 TblMin=1
 ▵Tbl=1
Indpnt: Auto  Ask
Depend: Auto  Ask
```

8. Press $\boxed{\text{2nd}}$ [TABLE] to view the first few rows of the table.

```
 X  │ Y₁  │
─────┼──────┼
 1  │2204 │
 2  │4032 │
 3  │5508 │
 4  │6656 │
 5  │7500 │
 6  │8064 │
 7  │8372 │
─────┴──────┴
X=1
```

✎ Go to the **Questions** section and answer #7.

Even though the window display shows only a few lines of the table, the blue arrow keys can be used to move up and down in the table.

9. Use $\boxed{\blacktriangledown}$ to move past the last line of entry in the table. The table will automatically scroll to permit this.

---

*The **Ask** feature of the **Indpnt** (independent) and **Depend** (dependent) variable options allows students to view "requested" values for each variable.*

(Note: The TI-80 does not have these features.)

*For example, if you set the independent variable to **Ask**, the dependent to **Auto**, and view the table, you will see a blank table. The calculator is "asking" you to provide a particular x value and it will then provide the corresponding y value. Students may want to explore this **Ask** option to see the different effects on the table for future reference.*

*Note: **TblStart** is used on the TI-83 rather than **TblMin**.*

✐ Go to the **Questions** section and answer #8 through #12.

Even though you probably did not do any computations, you were still doing mathematics as your group created arguments and explanations as to why the maximum volume has to occur between your two chosen values. Reasoning is also an important part of doing mathematics.

A way to verify your argument is to examine a table of values between your two whole number values. You need to redefine the table set-up values for *x* in order to do this.

1. Press [2nd] [TblSet].

2. Set **TblMin** to the smaller of the two numbers you argued for above.

3. Set Δ**Tbl** to .**1** so that the change between *x* values will now be *tenths* of a centimeter. Leave the other two lines at **Auto**.

4. Construct a new table by pressing [2nd] [TABLE].

Examine this table to determine the best choice for *x* now that *x* is restricted to tenths of a centimeter. In viewing the table you may find it helpful to highlight a value in the **Y1** column (use the blue right and left arrow keys to move back and forth between columns) and then move up and down the table (again with the blue arrow keys). Using these techniques will allow you to view the computed value of each **Y1** at the bottom of the screen rather than the rounded value shown in the table.

✐ Go to the **Questions** section and answer #13 through #16.

After changing the ΔTbl setting to 0.1 and viewing the table, many students want to explore "What happens if?" questions. This is exciting to see as they "zoom in" on a more precise solution. In fact, the precision is unrealistic if they think of having to cut out a square with side length 7.847 cm! Again, consider the mathematics that they are doing as they find this maximum volume. You are paving the way for the concepts of calculus!

Questions

1. If the class decided to cut 10-cm squares from the corners of the metal sheets, determine the dimensions and volume of the box that could be formed from the remaining sheet metal.

H= 10 cm
L= 40 cm
W= 20 cm
V= 8000 cm³

<div align="center">

Size of square: 10 cm

Height: _____cm Length: _____cm Width:_____cm

Volume: _____cm³

</div>

2. What would be the dimensions and volume of the box if 5-cm squares were cut from the corners of the metal sheets?

H= 5 cm
L= 50 cm
W= 30 cm
V=7500 cm³

<div align="center">

Size of square: 5 cm

Height: _____cm Length: _____cm Width:_____cm

Volume: _____cm³

</div>

As can be seen from the previous examples, cutting smaller squares does not necessarily result in a box with larger volume. Hopefully your students will chew on that idea for a while. Can they explain why that is so? Intuition may be that the volume should be larger. What is happening?

3. Does cutting out and throwing away less sheet metal necessarily result in a box with a larger volume? Explain.

_____

_____

_____

_____

4. Squares of many different sizes can be removed from the corners of the metal sheets. If cuts are made in whole centimeter lengths, the smallest square would have a side of length 1 cm.

The smallest dimension of the box, 40 cm, dictates the largest cut of 19 cm since 20 cm would result in a box with no volume.

What is the largest possible value for the length of the square? Explain why no larger value could be used.

_____

_____

_____

5. Ms. Hawkins decides she wants to have boxes made with the largest possible volume. How could the class determine which size square to cut out so that a box with the largest volume is produced?

_____

_____

_____

_____

A variety of answers may result, including making all possible boxes and determining which has the largest volume. Part of what mathematicians do is to analyze a problem to determine if there is a numerical or symbolic way to attack the problem that would minimize the time spent in carrying out the investigation.

✦ *Return to page 88.*

6. How would you calculate the volume of a storage box if you only knew the height of the box?

_____

_____

_____

We can calculate the values 60–2H and 40–2H to get the other two dimensions, length and width. The product of the three dimensions gives the volume.

✦ *Return to page 89.*

7. Write a sentence that would communicate the meaning of the two numbers in the first row of the table.

_____

_____

_____

When a square of 1 cm is cut out from each corner of the rectangular piece of metal, a box with a height of 1 cm and volume of 2204 cm³ can be produced.

✦ *Return to page 90.*

8. Describe any patterns you see in the sequence of volumes as you continue to scroll to the value **X=19**. For example, as the size of the cut squares increases, what happens to the volume of the box?

_____

_____

_____

_____

The volume increases as you scroll through the table to a maximum value of 8448 cm³ at x= 8 cm. After that, the volume decreases to a low of 836 cm³ at x = 19 cm. If you continue to scroll past 19 cm, you get zero and negative volumes. This doesn't make sense for the problem since 19 cm is the largest possible whole centimeter cut.

9. What size of square cut (value of *x*) gives a box with the largest volume (value of **Y1**)?

Height: _____cm Length: _____cm Width:_____cm

Volume: _____cm³

H = 8 cm, L = 44 cm, W = 24 cm, and
V = 8448 cm³

10. What are the dimensions and volume of the box that the industrial technologies class should make for Ms. Hawkins?

Same dimensions as those listed above.

Height: _____cm   Length: _____cm   Width:_____cm

Volume: _____cm^3

11. You should have found that the box of largest volume is made by cutting squares with sides of length 8 cm. However, this result assumes the height x is restricted to whole number values. What if we could now make cuts to tenths of a centimeter in the sheet metal? Between what two whole number heights do you believe the maximum volume would be found? Write down those two values along with their corresponding volumes.

X= 7 cm   V= 8372 cm^3
X= 9 cm   V= 8316 cm^3

X=_____   V=_____

X=_____   V=_____

The volume has been gradually increasing as x goes from 1 to 8. After x = 8 cm we see the volume starts to decrease. Since the relationship between x and y (height and volume) is not linear, we are picturing a "curve" in the data between 7 and 9. But we're not sure whether the top of this curve is between 7 and 8 or between 8 and 9.

12. Discuss these choices in your group and provide a convincing argument to show that the maximum volume must occur for some x between your two chosen values.

_____

_____

_____

_____

Students are sometimes surprised to see the maximum value occur where it does. They anticipate that it will perhaps be halfway between x=8 and x=9 or that it would occur after x=8 instead of before x=8.

13. Describe any surprises as you scrolled through the table of values.

_____

_____

_____

_____

x = 7.8 cm

14. What is the new best value for x?

_____

15. What is the volume of the best box?

V = 8448.132 cm³

_____

16. Is the volume of this box significantly larger than
the one with a height of 8 cm?

_The volume is not much
larger._

_____

_____

17. In summary for this numerical look at volume,
write three key points or ideas your group has
found while determining the maximum volume for
the boxes Ms. Hawkins needs for her physical
sciences class.

_"Sample" ideas: the
volume of boxes
increases and decreases
as you increase the
height of the box; we
were able to find the
volume of the box only
knowing one dimension,
the height; we were able
to write an equation
relating height and
volume; the relationship
between the height and
volume is not linear._

a. _____

_____

b. _____

_____

c. _____

_____

Activity 10: Box It Up (A Graphical Look) examines this
same problem using a graphical approach.