Critical Points and Local Extrema

Math Objectives

- Students will identify critical points using the definition.
- Students will identify local maxima and minima using the definition.
- Students will understand that local maxima and minima must occur at critical points but that not every critical point is the location of a local maximum or local minimum.
- Students will reason abstractly and quantitatively. (CCSS Mathematical Practice)
- Students will construct viable arguments and critique the reasoning of others. (CCSS Mathematical Practice)

Vocabulary

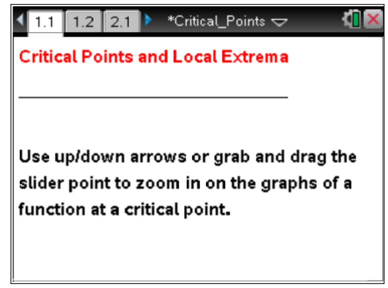
- critical point
- local maximum, minimum, extrema

About the Lesson

- This lesson involves visualizing the connections between the critical points and local extrema.
- As a result, students will:
  - Zoom in on function graphs at different types of critical points (including stationary points, locations of vertical tangents, “corners,” and cusps) to determine whether the slope of the tangent line is zero or undefined.
  - See that a local maximum or minimum occurs at critical points, but the examples illustrate that not every critical point is a local extremum.
  - Use the first derivative test as a means to identify local maximum and local minimum.
  - Build on their familiarity with the concept of the derivative at a point as the local slope of the function graph at that point.

TI-Nspire™ Navigator™ System

- Use Quick Poll to assess students’ understanding.
- Use Screen Capture to share students’ work.
- Collect student documents and analyze the results.
- Utilize Class Analysis to display students’ answers.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point
- Move a slider bar
- Grab and drag points along a graph
- Move between screen panes on a single page

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Materials:

Student Activity
Critical_Points_Student.pdf
Critical_Points_Student.doc

TI-Nspire document
Critical_Points.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

> **Tech Tip:** When on a page having multiple screens in its layout, ctrl tab will move from screen to screen. If the pointer arrow cursor is available, you can simply move to another screen pane using it, but you must click once inside the pane to make it active. An active screen pane has a bold outline around it.

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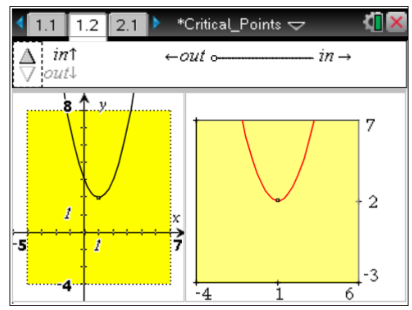

1. The graph of the differentiable function shown in the left window has a box centered around the point (1, 2). You can use the up/down arrows at the top of the screen (or drag the point on the line segment) to see a "zoomed in" view of this boxed area of the graph in the right window.

   a. This function has a local minimum at $x = 1$. Using the graph and the definition of local minimum above, explain why.

   <u>Sample answer:</u> The function has a local minimum at $x = 1$ because the function values for points near $x = 1$ are all greater than 2; or $x = 1$ is a local minimum because the graph "goes up" on either side of the point (1, 2), so the function value here is less than the function value at neighboring points.

   b. What appears to happen to the graph as you zoom in on the point (1, 2)?

   <u>Answer:</u> The graph appears to be a horizontal line.

   c. What is $\mathbf{f}'(1)$? Explain your answer. Why is $c = 1$ a critical point of **f?**

   <u>Answer:</u> $\mathbf{f}'(1) = 0$ because the slope of the horizontal tangent line is 0. Using the definition above, $c = 1$ is a critical point because $\mathbf{f}'(1) = 0$.

> **Teacher Tip:** If students have trouble connecting the ideas in 1b and 1c, you may wish to review the following: If a function **f** is differentiable at $x = a$, then its graph will appear to become linear as you zoom in on the point $(a, \mathbf{f}(a))$. The derivative $\mathbf{f}'(a)$ is the "local slope" of the graph of $y = \mathbf{f}(x)$ at the point $(a, \mathbf{f}(a))$.

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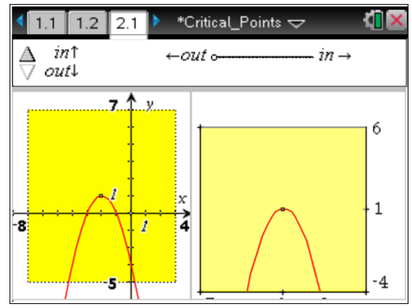

2. This is the graph of a function having a local maximum at $x = -2$.

   a. What appears to happen to the graph as you zoom in on the point (–2, 1)?

      <u>Answer:</u> The graph appears to be a horizontal line.

   b. What is the value of **f′**(–2)? Explain your answer. Why is $c = -2$ a critical point of **f?**

      <u>Answer:</u> **f′**(–2) = 0 because the slope of the horizontal tangent line is 0. Using the definition above, $c = -2$ is a critical point because **f′**(–2) = 0.

   c. What value could the derivative of a function have at the location of a local maximum or minimum? Explain your answer.

      <u>Sample answer:</u> Students may reason that the derivative must be zero at a minimum or maximum value because the graph will always "flatten out" when you zoom in at a "peak" or "low point." Or, the derivative must be zero because the tangent line would need to be horizontal at the local minimum and maximum points.

> **Teacher Tip:** Given the examples thus far, it is reasonable for students to conclude that the derivative must equal zero at a local maximum or minimum. The next two examples are used to illustrate that the derivative may not exist at a local minimum or maximum and thus just locating points for which the derivative of a function is equal to zero is not sufficient for finding local extrema.

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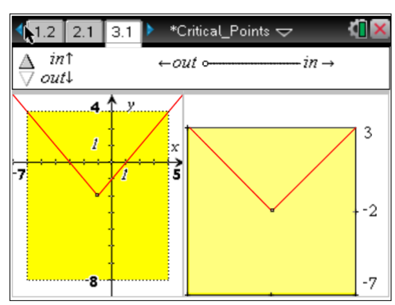

3. This is the graph of a function having a local minimum at $x = -1$.

   a. What happens to the graph as you zoom in on the point (–1, –2)?

      <u>Answer:</u> The graph does not appear to change as you zoom in. The V shape of the graph is maintained.

b. Assuming this behavior persists no matter how far you zoom in, is this function differentiable at $x = -1$? Why or why not?

Answer: The function is not differentiable at –1 because the graph does not become linear as you zoom in. (Instead, there are "two lines" with different slopes.)

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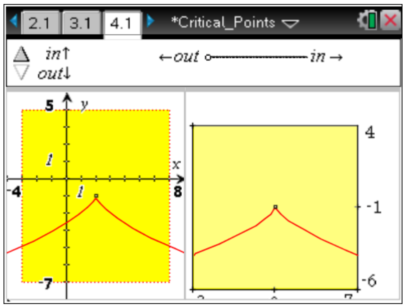

4. This is the graph of a function having a local maximum at $x = 2$.

a. What happens to the graph as you zoom in on the point (2, –1)?

Answer: The graph appears vertical, with both sections of the graph, left and right of the point (2, –1), moving closer together above the point (2, –1). (This is a cusp, where the slopes of the tangent lines approach positive and negative infinity as the critical point is approached from the left and right, respectively.)

b. Assuming this behavior persists no matter how far you zoom in, is this function differentiable at $x = 2$? Why or why not?

Answer: The function is not differentiable at $x = 2$ because the graph does not become linear when you zoom in (the slope of the tangent line would not exist at $x = 2$).

c. Do these examples support your answer to question 2c? If so, explain why. If not, how might you modify your previous answer?

Answer: The answer depends on students' answers to question 2c; most students will need to modify their response to conclude that it is also possible that the derivative of a function may not exist at a local maximum or minimum.

d. Could a function have a local maximum or minimum at a non-critical point? Why or why not?

Answer: Students should notice that in all of the examples, the local minimum and maximum points were also critical points of the function. An appropriate response should include some statement that the local minima and maxima of a function must occur at critical points. Reasoning that a change in the sign of the slope at a point would require the slope to be either 0 or undefined provides the grounding for the first derivative test.

Teacher Tip: CAUTION: It is important to note that a few examples do not constitute a proof of this statement. The examples given have provided only an intuitive or visual basis for this relationship.
This might be an appropriate time to formalize student conjectures by providing a statement of Fermat's theorem: if f has a local maximum or minimum at c, then c is a critical number of **f.**

TI-Nspire Navigator Opportunity: ***Quick Poll***
See Note 1 at the end of this lesson.

Teacher Tip: The purpose of the next two examples is to illustrate that not all critical points are local maximum or minimum points of a function. In other words, critical points must be tested.

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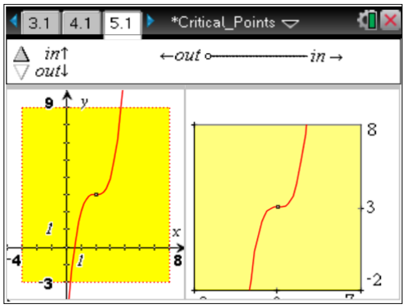

5. The graph of this increasing function has a horizontal tangent at the point $x = 2$.

   a. Is $x = 2$ a critical point? Why or why not?

   Answer: $x = 2$ is a critical point because $\mathbf{f}'(2) = 0$; when you zoom in on the point (2, 3), the graph appears horizontal.

   b. Does **f** have either a local minimum or local maximum at $x = 2$?

   Answer: No, $x = 2$ is neither a local minimum nor a local maximum, since the graph is strictly increasing everywhere.

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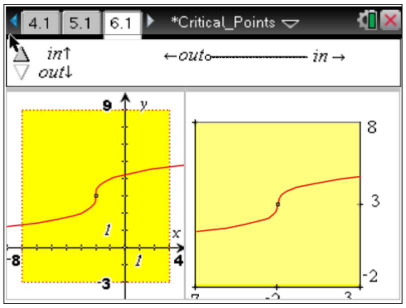

6. The graph of this increasing function has a vertical tangent at the point $x = –2$.

   a. Is $x = –2$ a critical point? Why or why not?

   Answer: $x = –2$ is a critical point because $\mathbf{f}'(2)$ is undefined; when you zoom in on the point (–2, 3), the graph appears vertical. The slope of the vertical tangent line is undefined.

TI-Nspire Navigator Opportunity: *Quick Poll* or *Screen Capture*
See Note 2 at the end of this lesson.

b. Does **f** have either a local minimum or local maximum at $x = -2$?

Answer: No, $x = -2$ is neither a local minimum nor a local maximum, since the function is increasing everywhere.

c. Does this contradict the statement you made in question 3d? Explain why or why not.

Answer: The answer depends on student responses to question 3d. This does not contradict the statement that local maxima and minima occur at critical points of a function. However, if students equated critical points with local extrema, they need to modify the statement to note that while it is true that all local maxima and minima of a function occur at critical points, not all critical points are local extrema.

Teacher Tip: If you introduced Fermat's theorem earlier, this is a good time to point out that the converse of this theorem is not a true statement.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- How to identify the critical points of a function given its graph.
- Why the local minima or maxima of a function occur at its critical points.
- Why not every critical point is a local minima or maxima.

Note: The assumption of continuity is another idea that should be addressed in the lesson wrap up. All of the functions presented in this lesson were continuous, a condition for the first derivative test. Although most functions students will encounter will be continuous, this is an important condition to note. Students could be challenged to consider what might happen if this condition were relaxed. Is it then possible for **f** to have a critical point a such that $\mathbf{f}(a)$ is a local minimum, but the derivative $\mathbf{f}'(x) > 0$ for $x < a$ and $\mathbf{f}'(x) < 0$ for $x > a$? If so, what might it look like?

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Note 1

Question 4, *Quick Poll*: This is a good point to send a Quick Poll on the following questions:

- Is every critical point a local maximum or minimum?
- Is every maximum or minimum found at a critical point?

Note 2

Question 6a, *Quick Poll or Screen Capture*: You may want to send a Quick Poll or collect a Screen Capture to verify that students understand critical points and local maximum and minimums.