Irrational Patterns

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Teacher Notes & Answers

7 8 **9** **10** 11 12

Scientific | Activity | Teacher | 60 min

Introduction

In this activity you will:

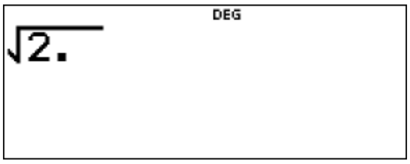
- Explore decimal approximations to *sequences* of *continued square roots.*
  - A ***sequence*** is an ordered list of numbers that follow a particular rule or pattern. Each number is a ***term*** of the sequence, called the first term, second term, third term ….
  - ***Continued square roots*** in these sequences will follow the rule
    $\text{term1} = \sqrt{a}$,Nextterm $= \sqrt{a + (\text{Previous term})}$ where $a = 2,3,4,5,6,\ldots$.
  - For example, if $a = 8$, then the sequence will be
    $\text{term1} = \sqrt{8}$, $\text{term2} = \sqrt{8 + (\text{term1})}$, $\text{term3} = \sqrt{8 + (\text{term2})}$, $\text{term4} = \sqrt{8 + (\text{term3})}$, …
- Look for patterns to help you predict which continued square roots will approximate an integer (whole number), when the rule is repeated many times.

Explore the sequence of approximations for $\left\{\sqrt{2}, \sqrt{2 + (\text{term1})}, \sqrt{2 + (\text{term2})}, \ldots\right\}$

Clear the history on the calculator screen.

Input $\sqrt{2.}$ and press <enter>.

Then input $\sqrt{2 + \text{answer}}$, and repeatedly press <enter>. (N.B. press <2nd> + <(-)> to input 'answer'.)



Keystrokes: TI-30X Plus MathPrint

Then

Question 1

a. What is the purpose of the decimal point in the input $\sqrt{2.}$?

It ensures that the output is a rational (decimal) approximation of the irrational number, $\sqrt{2}$

b. When you entered $\sqrt{2 + \text{ans}}$ and repeatedly pressed <enter>, what are some of the things you noticed about the outputs?

Answers will vary, but the key idea is that each time <enter> is pressed, the last answer replaces the answer before that, under the square root sign. This makes the process recursive.

c. Look at the sequence of approximations after the 17th output. Explain why these outputs appear as a whole number.

If the calculator mode setting is set to, say, 6 digits, then any value greater than 1.99999 will round to 2.00000. The calculator may display this as 2.

Author: F. Moya

The sequence of approximations that you created is a **recursive** sequence, meaning that the next term in the sequence is obtained from the previous term, using a rule: term1 $=\sqrt{2}.$,term2 $=\sqrt{2+\text{term1}}$,term3 $=\sqrt{2+\text{term2}}$,term4 $=\sqrt{2+\text{term3}},\ldots$

Explore other sequences of approximations $\{\sqrt{a},\sqrt{a+(\text{term1})},\sqrt{a+(\text{term2})},\ldots\}$

You will have noticed that for $a=2$, the continued square root approximates to an integer (whole number) when the rule Nextterm $=\sqrt{a+(\text{Previous term})}$ is repeated many times.

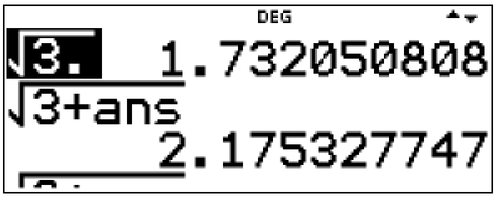
Question 2

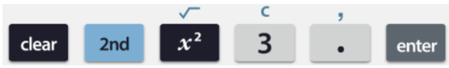
For $a=3$ to $a=6$, investigate which continued square roots also approximate an integer when the rule is repeated many times.

Continued square root term1 $=\sqrt{a}$ Nextterm $=\sqrt{a+(\text{Previous term})}$	Approximates an integer when rule repeated many times? Y/N	If yes, what value?
$a=3$	N	
$a=4$	N	
$a=5$	N	
$a=6$	Y	3

Clear the screen and repeat the previous steps.

For example, for $a=3$.





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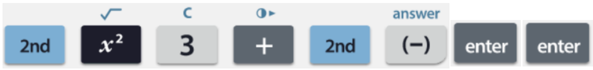


Question 3

a. Write down the first two values of a for which the continued square root approximates an integer when the rule is repeated many times.

When the rule is repeated many times: $\sqrt{2+(\text{previousterm})}\approx 2$ and $\sqrt{6+(\text{previousterm})}\approx 3$

b. Predict the next value of a whose continued square root also approaches an integer value. Use your calculator to check whether your prediction is correct.

Answers will vary.

Author: F. Moya

Speeding up the search for integer approximations $\{\sqrt{a}, \sqrt{a+(\text{term1})}, \sqrt{a+(\text{term2})}, \ldots\}$

You will have noticed that by the 4th term of the sequence it becomes clear whether the sequence of continued square root is approaching an integer.

A sequence of continued square roots with rule $\text{term1} = \sqrt{a}$, $\text{Nextterm} = \sqrt{a + (\text{previousterm})}$

could be written $\left\{\sqrt{a}, \sqrt{a+\sqrt{a}}, \sqrt{a+\sqrt{a+\sqrt{a}}}, \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}, \ldots\right\}$.

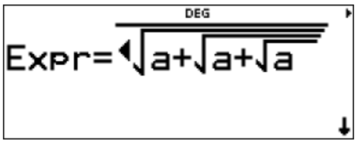
Question 4

Evaluate the decimal approximation of the 4th term with expression $\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$, for $a = 6$ to $a = 20$.

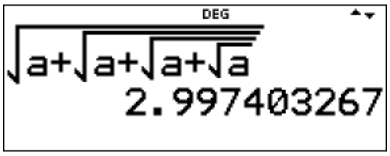
Record in the table the values of $a$for which the 4th term approximation is close to an integer.

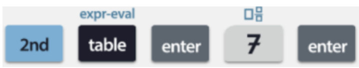
4th term $\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$	Approximates integer? Y/N	If Y, value?
$a = 6$	Y	3
$a = 7$	N	
$a = 8$	N	
$a = 9$	N	
$a = 10$	N	
$a = 11$	N	
$a = 12$	Y	4
$a = 13$	N	
$a = 14$	N	
$a = 15$	N	
$a = 16$	N	
$a = 17$	N	
$a = 18$	N	
$a = 19$	N	
$a = 20$	Y	5

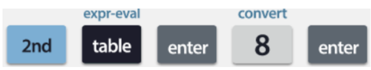
Clear the screen and repeat the following steps for $a = 6$ to $a = 20$.



Then press <enter>, input $a = 6$and <enter> again.



Then for $a = 7$, 

Then for $a = 8$, 

Repeat similar keystrokes for other values of a.

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Looking for patterns and making predictions

Question 5

a. Complete the table below for all values of a from 1 to 20, for which the sequences of continued square roots approximate to an integer, when the rule is repeated many times.

a	Integer value approached by the continued square roots when rule Nextterm $= \sqrt{a + (\text{Previous term})}$,term1 $= \sqrt{a}$, is repeated many times.
2	2
6	3
12	4
20	5
30	6
42	7
56	8

b. Describe any pattern you observe in the table.

Some possible patterns are shown. Students may describe their pattern(s) in words.

a	Repeat many times $\sqrt{a + (\text{previousterm})}$	Pattern 1	Pattern 2	Pattern 3
2	2	0 + 2 = 2	$1 \times 2 = 2$	$2^2 - 2 = 2$
6	3	2 + 4 = 6	$2 \times 3 = 6$	$3^2 - 3 = 6$
12	4	6 + 6 = 12	$3 \times 4 = 12$	$4^2 - 4 = 12$
20	5	12 + 8 = 20	$4 \times 5 = 20$	$5^2 - 5 = 20$
30	6	20 + 10 = 30	$5 \times 6 = 30$	$6^2 - 6 = 30$
42	7	30 + 12 = 42	$6 \times 7 = 42$	$7^2 - 7 = 42$
56	8	42 + 14 = 42	$7 \times 8 = 56$	$8^2 - 8 = 56$

c. Use the pattern(s) to extend the table by predicting the next three values of a for which the continued square roots approximate an integer. Use your calculator to check whether your predictions in the extended table are correct.

Shown in tables above.

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Extension

Question 6

For the patterns observed in the table for Question 5, suggest a reason for these patterns arising.

Answers will vary.
Sample answer. Let the n^{th}term $= x$, where n is a **very large** number. As students observed earlier, when the rule is repeated many times, the next term will also equal x.
Therefore Nextterm $= \sqrt{a + (\text{previousterm})}$ becomes $x = \sqrt{a + x}$.
$x = \sqrt{a + x}$ can be expressed as $a = x^2 - x$ or as $a = x(x - 1)$ etc., as observed in the patterns.

Question 7

Explore the sequence of continued square roots with rule term1 $= \sqrt{1}$,Nextterm $= \sqrt{1. + (\text{previousterm})}$. Write down the value of the 10th term rounded to four decimal places.
Use an internet search engine to research what is special about this number, and some examples of where this number arises in mathematics and in nature.

Answers will vary. When the rule is repeated many times, $\sqrt{1 + (\text{previousterm})} \approx 1.6180$. This number is known as the golden ratio.