The Trigonometric Derivative

ID: 9289

Time required
45 minutes

Activity Overview

In this activity, students will derive the derivative of the functions y = sin(x), y = cos(x),and y = tan (x), and work with the derivative of both y = sin(u) and y = cos(u). In the process, the students will determine that $\lim_{h\to 0}\frac{\cos(h)-1}{h}=0$.

Topic: Formal Differentiation

- *Use* ***Limit*** *to show* $\lim_{h\to 0}\frac{\sin(a+h)-\sin(a)}{h}=\cos(a)$ *and verify the rule for differentiating f(x)=sin(x).*
- *Use* ***Limit*** *to show* $\lim_{h\to 0}\frac{\cos(a+h)-\cos(a)}{h}=-\sin(a)$ *and verify the rule for differentiating f(x)=cos(x).*
- *Use* ***Derivative*** *to verify the Generalized Rules for differentiating the sine and cosine functions.*

Teacher Preparation and Notes

- *This investigation derives the definition of the derivative of y = sin(x), y = cos(x). The students should be familiar with keystrokes for the* ***Limit*** *command, the* ***Derivative*** *command, entering all the trigonometric functions, and setting up and displaying a table.*
- *The trigonometric identities for* $\sin(a+b)=\sin(a)\cdot\cos(b)+\sin(b)\cdot\cos(a)$ *as well as* $\cos(a+b)=\cos(a)\cdot\cos(b)-\sin(a)\cdot\sin(b)$ *are used in the derivation. In addition,* $\lim_{h\to 0}\frac{\sin(h)}{h}=1$ *will be used.*
- *The calculator should be set in radian mode before the derivatives are taken.*
- *Before starting this activity, students should go to the home screen and select* ***F6:Clean Up > 2:NewProb****, then press* [ENTER]*. This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.*
- ***To download the student worksheet, go to education.ti.com/exchange and enter "9289" in the keyword search box.***

Associated Materials

- *TrigonometricDerivative_Student.doc*

Problem 1 –Derivative of the Sine Function

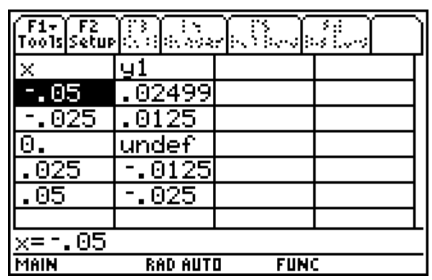
Students are to use the definition of derivative to set up the limit of sin(*x*). The identity for the sine of the sum of two angles $\sin(a+b)=\sin(a)\cdot\cos(b)+\sin(b)\cdot\cos(a)$ is needed to simply the limit. You may want to review the identity with students first.

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0}\frac{\sin(a+h)-\sin(a)}{h} && \text{def of derivative}\\
&= \lim_{h\to 0}\frac{\sin(a)\cdot\cos(h)-\sin(h)\cdot\cos(a)-\sin(a)}{h} && \text{substiute trig identity}\\
&= \lim_{h\to 0}\frac{\sin(a)\cdot(\cos(h)-1)-\sin(h)\cdot\cos(a)}{h} && \text{combine like terms}\\
&= \lim_{h\to 0}\frac{\sin(a)\cdot(\cos(h)-1)}{h}+\lim_{h\to 0}\frac{\sin(h)\cdot\cos(a)}{h} && \text{split into two limits}
\end{aligned}
$$

In the first limit, $\lim_{h\to 0}\frac{\cos(h)-1}{h}$ needs to be determined since substitution gives $\frac{0}{0}$.

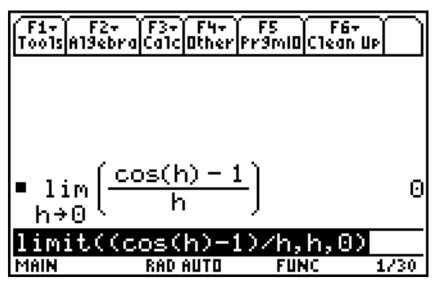
For the second limit, recall that $\lim_{h\to 0}\frac{\sin(h)}{h}=1$.

Students will then use the **table set** command to generate a table of values for $y=\frac{\cos(x)-1}{x}$. with x = -0.1 and Δx = 0.025.



The undefined at x = 0 reflects that the ratio has a zero in the denominator. However, the table indicates that values are tending toward zero.

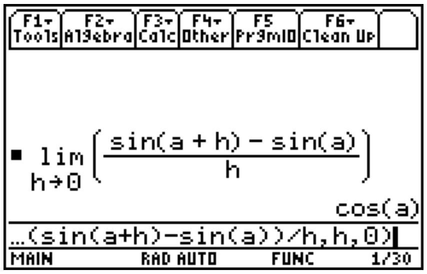
Students will support this conclusion by using the **Limit** command **F3:Calc>3:Limit**.

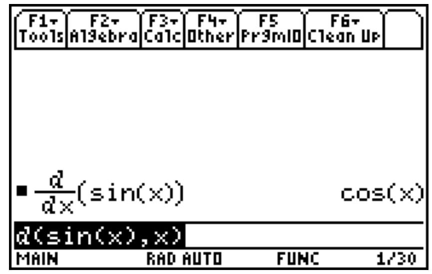


Now students will substitute $\lim_{h\to 0}\frac{\cos(h)-1}{h}=0$ and $\lim_{h\to 0}\frac{\sin(h)}{h}=1$ into the limit they set up earlier. So,

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0}\frac{\sin(a+h)-\sin(a)}{h}\\
&= \lim_{h\to 0}\frac{\sin(a)\cdot(\cos(h)-1)}{h}+\lim_{h\to 0}\frac{\sin(h)\cdot\cos(a)}{h}\\
&= \sin(a)\cdot 0+1\cdot\cos(a)\\
&= \cos(a)
\end{aligned}
$$

To confirm this, student can use the **Limit** command and **Derivative** command.



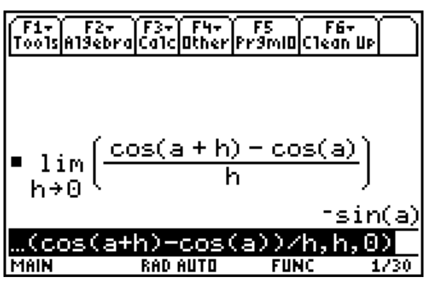


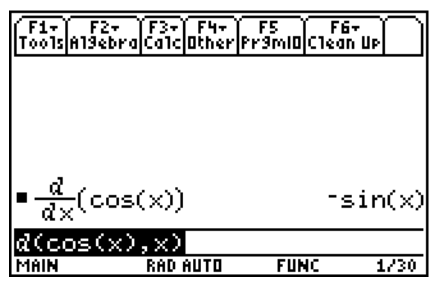
Problem 2 – Derivative of the Cosine Function

In this problem, students will use the definition of derivative to set up a limit to find the derivative of $f(x) = \cos(x)$. They will need to use the trigonometric identity
$\cos(a+b) = \cos(a)\cdot\cos(b) - \sin(a)\cdot\sin(b)$.

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0}\frac{\cos(a+h)-\cos(a)}{h}\\
&= \lim_{h\to 0}\frac{\cos(a)\cdot\cos(h)-\sin(a)\cdot\sin(h)-\cos(a)}{h}\\
&= \lim_{h\to 0}\frac{\cos(a)\cdot\left(\cos(h)-1\right)-\sin(a)\cdot\sin(h)}{h}\\
&= \lim_{h\to 0}\frac{\cos(a)\cdot\left(\cos(h)-1\right)}{h}-\lim_{h\to 0}\frac{\sin(a)\cdot\sin(h)}{h}\\
&= \cos(a)\cdot 0 + 1\cdot\sin(a)\\
&= -\sin(a)
\end{aligned}
$$

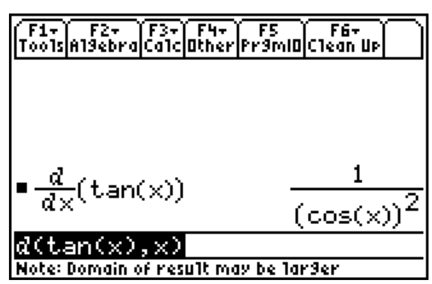
Students can verify their answers by using the **Limit** command and **Derivative** command.





Problem 4– Derivative of the Tangent Function

In this problem, students will write $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and use the quotient rule to find the derivative of tangent:



$$\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)=\frac{\cos(x)\bullet\frac{d}{dx}(\sin(x))-\sin(x)\bullet\frac{d}{dx}(\cos(x))}{\cos^2(x)}$$

$$=\frac{\cos(x)\bullet\cos(x)-\sin(x)\bullet(-\sin(x))}{\cos^2(x)}$$

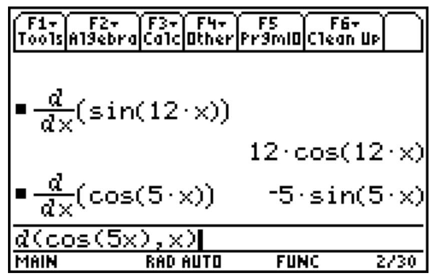
$$=\frac{\cos^2(x)+\sin^2(x)}{(\cos(x))^2}$$

$$=\frac{1}{\cos^2(x)}$$

Students will confirm this result using the Derivative command. Written in terms of the reciprocal function, the derivative of the tangent is $\frac{dy}{dx}\tan(x)=\sec^2(x)$.

Extension – Derivative of *y* = sin(*u*) and *y* = cos(*u*)

Students are to use the chain rule for both the sine and cosine functions. They can use the **Derivative** command for the first set and it will give the result at the right.



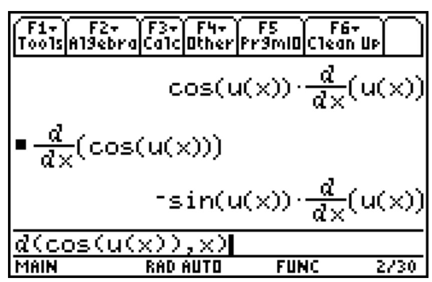
$$y=\sin(u)\rightarrow\frac{dy}{dx}=\cos(u)\cdot\frac{du}{dx}$$

$$y=\cos(u)\rightarrow\frac{dy}{dx}=-\sin(u)\cdot\frac{du}{dx}$$

$$f(x)=\sin(12x)\rightarrow f'(x)=12\cos(12x)$$

$$g(x)=\cos(5x)\rightarrow g'(x)=-5\sin(5x)$$

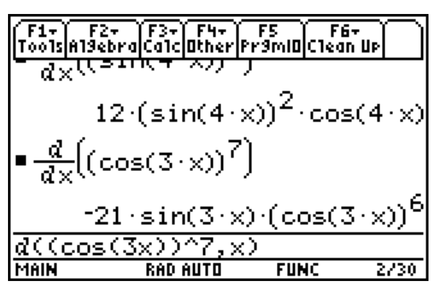
Students are to then use the **Derivative** command for sin(*u*) and cos(*u*). Remind them to use *u*(*x*) for *u*.



$$\frac{d}{dx}\sin(u(x))=\cos(u(x))\cdot\frac{d}{dx}u(x)$$

$$\frac{d}{dx}\cos(u(x))=-\sin(u(x))\cdot\frac{d}{dx}u(x)$$

Finally, students will find the derivative of the functions below. Remind them to use the parentheses so that the trigonometric function is raised to the power.



$$h(x)=\sin^3(4x)\rightarrow h'(x)=12\sin^2(4x)\cdot\cos(4x)$$

$$j(x)=\cos^7(3x)\rightarrow j'(x)=-21\sin(3x)\cdot\cos^6(3x)$$