EXPLORATIONS™

Activity 5

Objectives

- Use the CellSheet™ App to determine geometric ratios of areas
- Find a quadratic function that models geometric ratios

Ratio of Areas

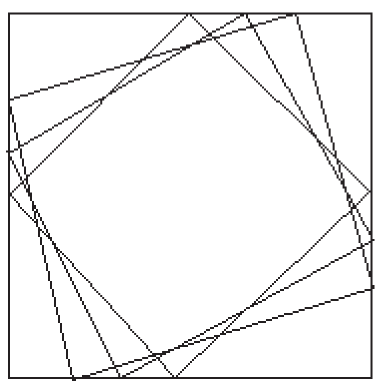
Introduction

The Ancient Greeks were the first to formulate many of the geometric relationships you study now. For example, they determined that the area of a triangle is half the area of a rectangle. They also formulated many of the trigonometric relationships that are so useful to builders, engineers, and sailors.

In this activity, you will determine the ratios of areas.

Problem

You have been asked to design an invitation for the school dance out of a 15 cm-by-15 cm square. You would like to fold over each corner of the square so that a small inner square is formed in which you will have the class year inscribed.



If you fold in from the midpoint of each side, the four corners meet and there is no open space in the middle. How far from the corner should each fold line begin and end if the flaps (the total shaded area) are to equal 40% of the total area of the paper?

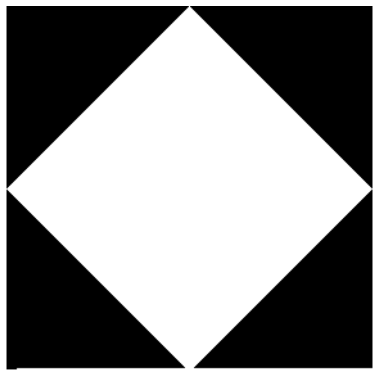


Figure 1

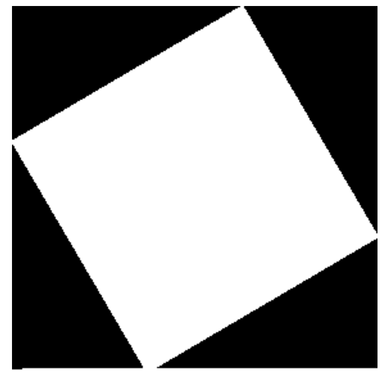


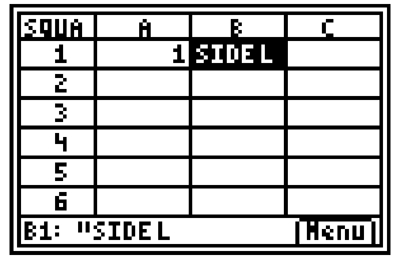
Figure 2

In Figure 1, the vertices of the smaller square are at the midpoints of the four sides of the larger square. The shaded area consists of four congruent triangles. No matter what the length of the sides of the large square, the shaded area represents 50% of the total area. As you move the corners of the small square, the shaded area decreases. (See Figure 2.)

To simplify calculations, let the lengths of the sides of the larger square equal 1.

Exploration

1. Start a new spreadsheet in the CellSheet™ App and name it **SQUARES**.
2. Enter **1** in cell A1. This will represent the length of each side of the larger square.
3. Type **"SIDE L** in cell B1 to indicate that **1** is the measure of the side of the larger square.
4. Type **"POINT** in cell B2, **"AREA** in cell C2, and **"RATIO** in cell D2.

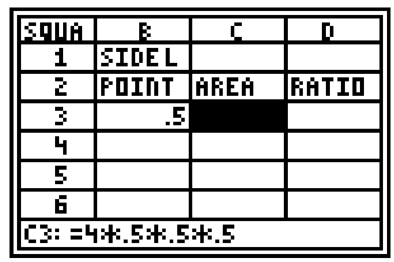


5. Enter **0.5** in cell B3. This represents the height and base of the shaded triangles. (They are isosceles triangles.)
6. To calculate the shaded area, enter the formula for the area of a triangle $\times$ 4 in cell C3.

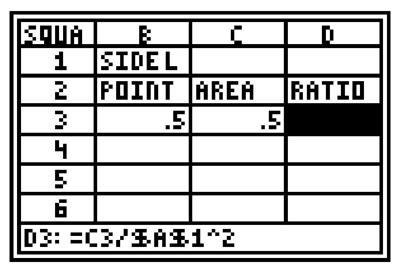
   Formula for the area of the triangle:

   ____

   Remember that you want to compare the area of the shaded area to the area of the larger square. The area of the shaded area is in cell C3. The area of the larger square is the value of cell A1 squared.



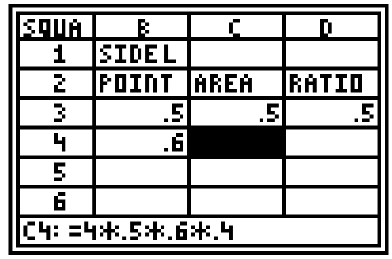
7. To determine the ratio of shaded area to the larger square, enter the formula **=C3/$A$1^2** in cell D3.



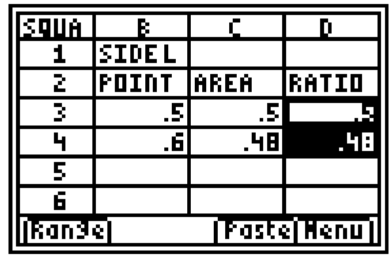
   Imagine rotating the smaller square so that each vertex is at the 0.6-point of the larger square (see Figure 2). The area of each of the four shaded areas (still congruent triangles) is still $\frac{1}{2} \times b \times h$, but the dimensions have changed to 0.4 and 0.6.

   Write the formula for the area of these triangles. ________________________.

8. Type **0.6** in cell B4. Enter the formula for the area of the triangle × 4 in cell C4.

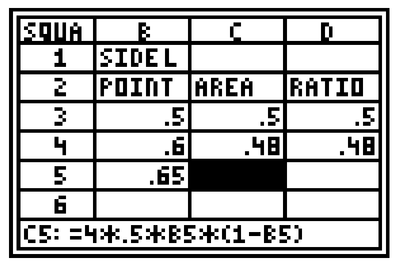


9. Copy the formula from cell D3 into cell D4. (The area of the larger triangle is constant.)

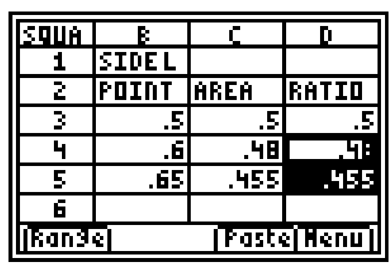


Before continuing, look for a formula to calculate the area of the shaded triangles as we change the dimensions. One formula is $\frac{1}{2} \times$ (point) × (1 – point) × 4.

10. Enter **0.65** in cell B5 and the formula **=4*.5*B5*(1–B5)** in cell C5.

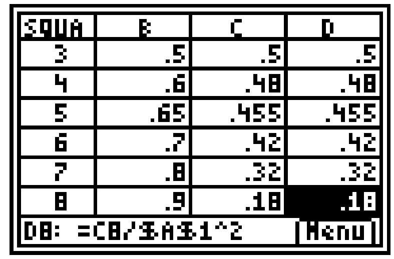


11. Copy the formula from cell D4 into cell D5.



Look closely at the results you got using this formula for the first time to check that they are realistic. You would expect the ratio to continue to decrease. You would expect the ratio not to change as much from 0.6 to 0.65 as it did from 0.5 to 0.6. It is a good practice to consider initial results before using a model for more calculations.

12. Type **0.7**, **0.8**, and **0.9** in cells B6 through B8. Copy the formulas in cells C5 and D5 through cells C8 and D8.

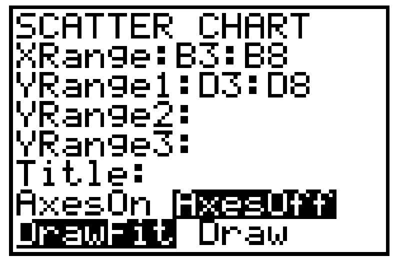


SQUA	B	C	D
3	.5	.5	.5
4	.6	.48	.48
5	.65	.455	.455
6	.7	.42	.42
7	.8	.32	.32
8	.9	.18	.18

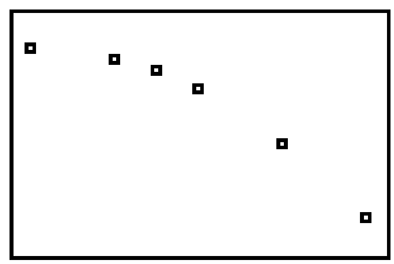
D8: =C8/$A$1^2 Menu

Look at the values in columns C and D. What do you notice about these values? What explanation can you give to explain these values?

Look at the data graphically. A scatter plot of the data is a good way to show the relationship between the length of the side of the triangle and the ratio of the two areas.



What do you notice about the graph? What does the graph of the data resemble?



The graph is in the form of a parabola, which means the data points are a quadratic function, $y = ax^2 + bx + c$, where x is the point and y is the ratio of the two areas. In this quadratic function, if x equals zero, then y must equal zero. If $x = 0$, the vertices of the smaller square are the same as those of the larger square, and there is no shaded area. So, $0 = a(0) + b(0) + c$, and $c = 0$.

In order to find the quadratic equation that fits the data, you will need to find the value of a and b for any two ordered pairs by using sets of data points in two quadratic equations $y = ax^2 + bx$, and solve for the variables.

13. a. Take any two ordered pairs from the data. For example, let's use (0.5, 0.5) and (0.6, 0.48).

$$x = 0.6 \text{ and } y = 0.48 \rightarrow 0.48 = 0.36a + 0.6b$$

$$x = 0.5 \text{ and } y = 0.5 \rightarrow 0.5 = 0.25a + 0.5b$$

Solve the system of equations for a.

$$\begin{array}{r} 0.5(0.48 = 0.36a + 0.6b) \\ 0.6(0.5 = 0.25a + 0.5b) \\ \hline 0.06 = {}^-0.03a \\ {}^-2 = a \end{array}$$

b. Substitution shows that $b = 2$. Thus, the quadratic equation is $y = {}^-2x^2 + 2x$.

Is the quadratic function found using those two ordered pairs the same that would be found using two different ordered pairs?

One way to answer this would be to try other ordered pairs. Another way is to see whether that quadratic model fits the data.

14. Type the quadratic model `=-2*B3^2+2*B3` in cell E3. Notice cell B3 is the x-value.

SQUA	C	D	E
3	.5	.5	
4	.48	.48	
5	.455	.455	
6	.42	.42	
7	.32	.32	
8	.18	.18	

E3: =-2*B3^2+2*B3

15. Copy the formula for the model down column E.

SQUA	C	D	E
3	.5	.5	.5
4	.48	.48	.48
5	.455	.455	.455
6	.42	.42	.42
7	.32	.32	.32
8	.18	.18	.18

E8: =-2*B8^2+2*B→ Menu

The model fits the data precisely! You have found a quadratic equation that models the relationship between the length of the sides of the triangles and the ratio of the area of the larger square and the shaded areas.

Student Worksheet

Name ________________________

Date ________________________

Reviewing Concepts

1. The formula for the area of a triangle is _____________________ . In this problem, the four triangles of the shaded area are ______________. Thus, the total area is _______________ times the area of any one of them. Because the triangles are also _______________ triangles, the two legs represent the ______________ and ________________.

Solving the Problem

2. The fold lines on the large square should be at ________________% if the total shaded area is to equal 40% of the total area of the paper (round to two decimal places).

3. After the four folds are made, the sides of the resulting square are _______ cm.

Analyzing the Data

4. Complete the table shown.

Point	*The Area*[1] (formula)	*The Area* (result)	The Ratio[2]
0.5			
0.6			
0.65			
0.7			
0.8			
0.9			

[1] The area is the shaded area, outside the smaller square and inside the larger square.

[2] The ratio is the ratio of The Area (result) to the area of the larger square.

5. As the smaller square is rotated clockwise, the total shaded area _________. The area approaches zero as the vertices of the smaller square get closer to those of the larger square.

6. The problem is seeking a model to predict the ratio of the area outside the smaller square (*S*) to the area of the larger square (*L*): *S*/*L*.

Because ________________, then $\frac{S}{L} = S$.

7. Make a sketch of the 0.6-point situation, and label the legs of each triangle.

8. Use the two points (0.6, 0.48) and (0.8, 0.32) to find the values for *a* and *b* in the quadratic model in the form $y = ax^2 + bx + c$, where $c = 0$. Are the values for *a* and *b* the same as those found in the Exploration?

Extending the Activity

1. The problem was simplified by using 1 as the side length of the larger square. Would the same model hold if the dimensions of the larger square were different? Explain your reasoning.

2. Now let the side of the larger square be 2. Create a spreadsheet to check your thinking.

Teacher Notes

Activity 5

Ratio of Areas

Objectives

- Use the CellSheet™ App to determine geometric ratios of area
- Find a quadratic function that models geometric ratios

Materials

- TI-84 Plus/TI-83 Plus

Time

- 60 minutes

Preparation

This activity requires students to imagine a square inside another square, with all four vertices of the smaller square on sides of the larger square. If you can show an animation of this event, either by finding an applet on the Internet, or by using Cabri® Jr., it may help students visualize the task and see the ratio of the areas before doing the calculations. If that is not possible, have students sketch the 0.5-point and then the 0.6-point squares as accurately as possible.

Elicit Questions

Have students cut out a square on graph paper. They will first fold in the four corners so they meet in the middle. Next, have them mark a point on the side equal to 60% of the length of the side and fold from there, comparing the area of the flaps. This will give students a visual to help them understand the Exploration activity more clearly.

Management

Students should be encouraged to consult with one another as they work through the Exploration. They can discuss their findings and their emerging hypotheses to explain their findings.

The problem may be solved a number of ways. Using the CellSheet App, students can change the value in cell B7 until the value in C7 gets close enough to 0.40. Students may also use the quadratic model to find the solution: $0.40 = {}^{-}2x^2 + 2x$ to find $x = 0.7236$. To find the measurements of the square after the folds are made, students may actually construct the square and find an approximate measurement. They may also use the Pythagorean Theorem to find the length of the hypotenuse of one shaded region.

Answers to Exploration Questions

6. $0.5 \times b \times h$ or $0.5 \times 0.5 \times 0.5$

7. $0.5 \times 0.6 \times 0.4$

12. Answers may vary, but students should note that the values in the two columns are the same. The ratio of the shaded area to the area of the larger square is the area over 1, which equals the area.

Answers to the Student Worksheet

Reviewing Concepts

1. $\frac{1}{2} \times b \times h$; congruent; 4; right; base; height

Solving the Problem

2. 75%

3. 11.6 cm

Analyzing the Data

4.

Point	*The Area*[1] (formula)	*The Area* (result)	The Ratio[2]
0.5	*= 4 × ½ × 0.5 × 0.5*	0.5	0.5
0.6	*= 4 × ½ × 0.6 × 0.4*	0.48	0.48
0.65	*= 4 × ½ × 0.65 × 0.35*	0.455	0.455
0.7	*= 4 × ½ × 0.7 × 0.3*	0.42	0.42
0.8	*= 4 × ½ × 0.8 × 0.2*	0.32	0.32
0.9	*= 4 × ½ × 0.9 × 0.1*	0.18	0.18

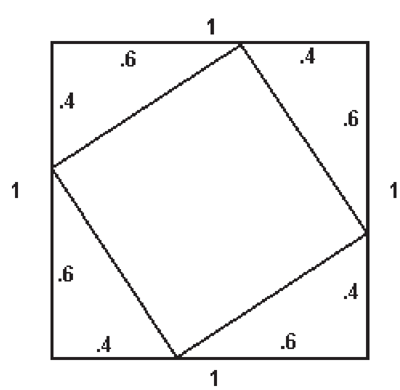
[1] The area is the shaded area, outside the smaller square and inside the larger square.

[2] The ratio is the ratio of The Area (result) to the area of the larger square.

5. Decreases

6. $L = 1$

7.



8. Substitute the values in the quadratic.

$0.48 = 0.36a + 0.6b$
$0.32 = 0.64a + 0.8b$

$0.8\,(0.48 = 0.36a + 0.6b)$
$0.6\,(0.32 = 0.64a + 0.8b)$

Use elimination

$0.384 = 0.288a + 0.48b$
$0.192 = 0.384a + 0.48b$
${}^{-}0.192 = 0.096a$
$a = {}^{-}2, \quad b = 2$

Yes, the results are the same.

Extending the Activity

1. The size of the larger square does not matter because the answer is a ratio of the two areas. If the size of the larger square were to increase, the size of the smaller square would also increase, keeping the ratio the same.

2. The value in cell A1 represents the length of each side of the larger square. By changing the contents of that cell to 2, the remaining calculations will be done using 2 as length of the side of the larger square. The values in column B will need to be multiplied by 2.

Other formulas will have to be changed to reflect the change in cell A1. The formula in cell C3 should be **=4*.5*(B3*$A$1)*(1–B3)*$A$1**.

B3*$A$1 represents one leg of the triangle and **(1–B3)*$A$1** represents the other.