EXPLORATIONS

Chapter 1

Recovering a Function from its Derivative: A Graphical Approach

If you know the derivative of a function, can you recover the function itself? Sometimes this is a simple task, other times it is difficult without the help of technology. The TI-86 can help you visualize a function when the only information you have about the function is an equation describing its derivative and a single point on the function. Since this equation contains a derivative, it is called a *differential equation*. The x and y coordinates of the point are called *initial conditions* for the function. The function you are looking for is called the *solution* to the differential equation. In this chapter, you will use the TI-86 to visualize solutions to differential equations with several examples. You will start with simple differential equations, so the solutions should look familiar. This should allow you to focus on technical skills. Later, you will investigate ways to use these technical skills to solve problems that would be difficult to solve without technology.

Resetting the TI-86

You need to select the factory default settings for your TI-86 before you do the first few examples in this chapter. Follow these steps to reset your TI-86 to the default settings.

1. Press [ON] [CLEAR] [2nd] [MEM] to see the MEM (MEMORY) menu. (Figure 1.1)

Figure 1.1

2. Press [F3] (**RESET**). (Figure 1.2)

   You can tell you are in the RESET submenu because **RESET** is highlighted on the top row.

Figure 1.2

3. Select the defaults feature by pressing [F3] (**DFLTS**). (Figure 1.3.)

Figure 1.3

4. Press [F4] to confirm selection of the default settings. (Figure 1.4.)

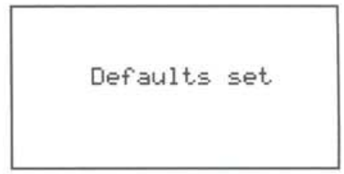

Figure 1.4

Example 1: Visual Solution to a Simple Differential Equation

If

$$\frac{dy}{dx} = x$$

and $y = 0$ when $x = 0$, what does the graph of $y = f(x)$ look like?

Solution

In this example,

$$\frac{dy}{dx} = x$$

is the differential equation and $y = f(x)$ is the solution to the differential equation. The initial conditions for $y = f(x)$ are $x = 0$, $y = 0$.

You can see the graph of $y = f(x)$ by following these steps:

Select Differential Equation Graphing Mode

Place the TI-86 in differential equation graphing mode by pressing [2nd] [MODE] and selecting **DifEq**. (To select **DifEq**, use the cursor movement keys to highlight **DifEq** and then press [ENTER]). (Figure 1.5)

Figure 1.5

Enter the differential equation

1. You will enter the differential equation on a screen called the differential equation editor (the **Q'(t) =** screen), which is selected from the GRAPH menu. Press [GRAPH] to see the GRAPH menu. (Figure 1.6)

Figure 1.6

2. Since **Q'(t) =** appears just above the $\boxed{\text{F1}}$ key, press $\boxed{\text{F1}}$ to access the **Q'(t) =** screen (the differential equation editor). (Figure 1.7)

   If the screen is not empty like the one in Figure 1.7, delete lines by pressing $\boxed{\text{F4}}$ (**DELf**) until your screen looks like Figure 1.7.

Figure 1.7

3. In differential equation graphing mode, you will use **t** in place of **x** and **Q** in place of **y.** Since the TI-86 uses the variables **t** and **Q**, enter the differential equation

$$\frac{dy}{dx} = x$$

   as **Q'1=t**. Press $\boxed{\text{F1}}$ to enter. (Figure 1.8)

Figure 1.8

Select the viewing format

1. The TI-86 offers several formats to view the solution to a differential equation. These are found in the format screen, which is selected from the GRAPH menu. Since you are currently in the differential equation editor (**Q'(t) =** screen), you can return to the GRAPH menu by pressing $\boxed{\text{EXIT}}$. (Figure 1.9)

Figure 1.9

2. Notice that **FORMT** does not appear in the GRAPH menu shown in Figure 1.9. There is, however, an arrow in the last box of the menu line that indicates there are more items available. Press $\boxed{\text{MORE}}$ to see these. (Figure 1.10)

Figure 1.10

3. Press $\boxed{\text{F1}}$ (**FORMT**) to select the format screen. (Figure 1.11)

Figure 1.11

4. The default settings are highlighted. They will be discussed later, but for now select **FldOff** by moving the cursor to **FldOff** and pressing $\boxed{\text{ENTER}}$. Make sure the highlighted entries on your screen match Figure 1.12.

Figure 1.12

Select the viewing window

1. *The viewing window is selected on the window editor,* which is selected on the GRAPH menu. Since you are currently in the GRAPH menu, press [F2] (**WIND**) to display the window editor. (Figures 1.13 and 1.14)

   You can see the values in Figure 1.14 by scrolling down the window editor with the cursor movement keys. These are the TI-86 default values for the this screen. You will not change these values at this time, but an explanation of the role of the window variables is in order.

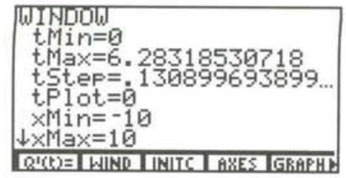

Figure 1.13

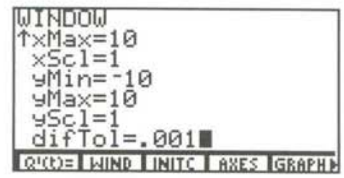

Figure 1.14

Variable	Function
tMin	The point at which the TI-86 will begin graphing the solution to the differential equation
tMax	The point at which the TI-86 will stop graphing the differential equation
tStep	The spacing between the points calculated in the solution
tPlot	The point at which plotting begins (Ignored when **t** is an axis)
xMin	The left boundary of the viewing window
xMax	The right boundary of the viewing window
xScl	The spacing between the marks on the x axis
yMin	The bottom boundary of the viewing window
yMax	The top boundary of the viewing window
yScl	The spacing between the marks on the y axis
difTol	Affects the speed and accuracy of the solution

Select initial conditions

1. Press [F3] (**INITC**). (Figure 1.15)

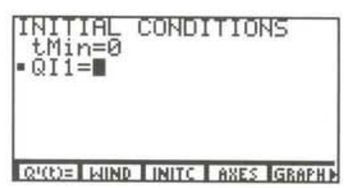

Figure 1.15

2. Since **x = t**, the initial value of **x** is stored in **tMin**. The default value of **tMin** is zero. Since **y = Q** the initial value of **y** is stored in **QI1**. Press [0] to enter the initial value for **QI1**. (Figure 1.16)

   You may have other **QI** variables in the initial conditions editor (**INITC** screen). They can be ignored since they should not affect the solution. The only **QI** variables that affect the solution are the ones with a square beside them.

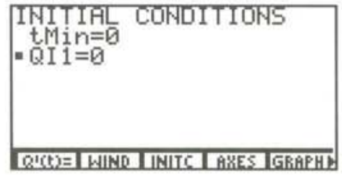

Figure 1.16

Select the axes

Select the variables to be used for the x and y axes with the axes editor, which is selected from the GRAPH menu. Press [F4] (**AXES**). (Figure 1.17)

The default settings are **x = t** and **y= Q**. Since this is what you want, you will not make any changes in the axes editor.

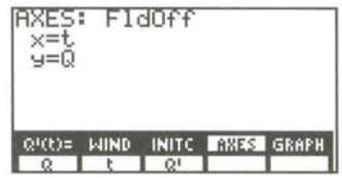

Figure 1.17

Graph the solution

1. What should the graph of the solution look like? Press [EXIT] to return to the GRAPH menu and then press [F5] (**GRAPH**) to see the solution to the differential equation. (Figure 1.18)

   Did you expect to see a parabola? The analytic solution to the initial value problem

$$\frac{dy}{dx} = x\,,\ x = 0,\ y = 0,$$

   is

$$y = \frac{x^2}{2}.$$

   You see only half the parabola in Figure 1.18 because the TI-86 begins graphing the solution at the point specified by the initial conditions. In this case, the initial point was (0,0), so the TI-86 began graphing at the origin.

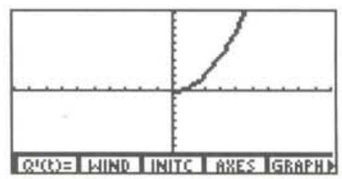

Figure 1.18

2. You can overlay the graph of

$$y = \frac{x^2}{2}$$

   on the solution shown in Figure 1.18 with the **DrawF** (Draw Function) command. This command is found in the GRAPH DRAW menu. Press [MORE] [F2] to select the DRAW menu. (Figure 1.19)

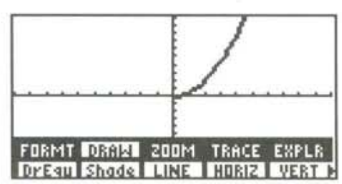

Figure 1.19

3. Press [MORE] [F2] to paste the **DrawF** command to the home screen. Now enter the expression x^2/2. (Figure 1.20)

   Even though the differential equation mode uses **t** and **Q** as graphing variables, the **DrawF** command uses **x** and **y**.

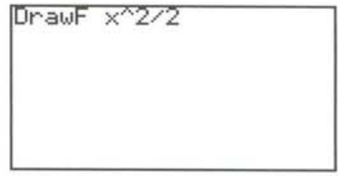

Figure 1.20

4. When you press [ENTER], you should see the graph of

$$y = \frac{x^2}{2}$$

   drawn on the same screen with the solution that you found using **DifEq** mode. (Figure 1.21)

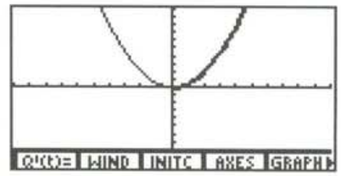

Figure 1.21

The graphing style used by **DrawF** results in a thinner curve than the solution drawn using differential equation mode. These different styles illustrate the fact that the differential equation graph began at the initial values $x = 0$, $y = 0$. In the next example, you will see the effect of changing the initial values.

Example 2: Changing Initial Values

Find the solution to the differential equation

$$\frac{dy}{dx} = x\,,\ x = {}^{-}2,\ y = 0.$$

This is the same problem as Example 1, but with a different initial value for x.

Solution

1. Change the initial conditions with the initial conditions editor. Press [GRAPH] [F3] (**INITC**) to select this menu. (If you are still in the GRAPH window, you only need to press [F3].) Use the cursor movement keys to move from one line to another and enter the values shown in Figure 1.22.

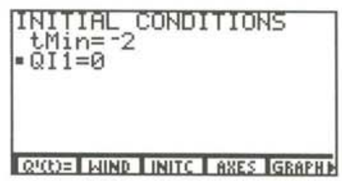

Figure 1.22

   Changing **tMin** in the initial conditions editor also changes the value of **tMin** in the window editor.

   Recall that **tMin** is the initial value of x and **QI1** is the initial value of y. How will these new initial conditions change the graph of the solution? Sketch your prediction.

2. Press [F5] (**GRAPH**) to see the graph. (Figure 1.23)

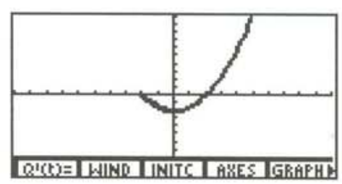

Figure 1.23

   The analytic solution is

$$y = \frac{x^2}{2} - 2\,.$$

   The TI-86 began graphing this parabola at the point ($^{-}$2,0).

3. What will the graph look like if you change **tMin** to $^-3$ and **tMax** to 4? Sketch your prediction and then make the necessary changes in the window editor and see if you were right. (Figures 1.24 and 1.25.)

   Figure 1.25 shows the graph of the solution beginning at **tMin** = $^-3$ and ending at **tMax** = 4. This graph corresponds to the initial values $x = {}^-3$, $y = 0$. In the next example, you will see how to enter several different sets of initial values at the same time.

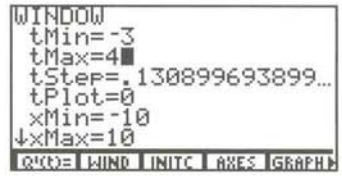

Figure 1.24

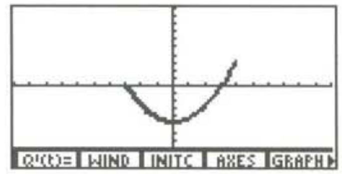

Figure 1.25

Example 3: Entering Several Initial Values as a List

Graph the solutions to the differential equation

$$\frac{dy}{dx} = x\,,$$

corresponding to the following sets of initial values:

$$x = {}^-2,\ y = {}^-5$$

$$x = {}^-2,\ y = 0$$

$$x = {}^-2,\ y = 5.$$

Solution

Use the same viewing window as Example 2, but change the initial values. You can enter several sets of initial values by using TI-86 lists. In this example, you will enter the initial values of y with the list { $^-5$, 0, 5}.

1. Press [GRAPH] [F3] (**INITC**) to select the initial conditions editor.

2. Press [2nd] [LIST] to display the brace symbols { }. (Figure 1.26)

Figure 1.26

3. Move the cursor to **tMin** and enter $^-2$. Then move the cursor to **QI1** and press [F1] [(-)] [5] [,] [0] [,] [5] [F2] to enter the list. (Figure 1.27)

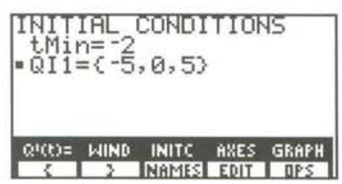

Figure 1.27

4. What graph(s) will this set of initial values produce? Press [2nd] [M5] (**GRAPH**) to find out. (Figure 1.28)

   The solution corresponding to each set of initial values appears to be a vertical shift of the first solution. In the next example you will see how slope fields illustrate the family of solutions corresponding to all possible sets of initial values.

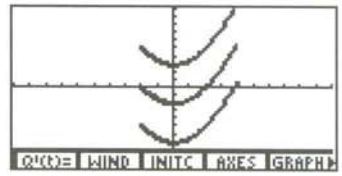

Figure 1.28

Example 4: Slope Fields

Graph the slope field associated with the differential equation

$$\frac{dy}{dx} = x.$$

Solution

Keep the same viewing window and list of initial conditions used in Example 3 and change the format to **SlpFld**.

1. Press [MORE] [F1] (**FORMT**), move the cursor to highlight **SlpFld**, and press [ENTER]. (Figure 1.29)

Figure 1.29

2. Press [F5] (**GRAPH**). (Figure 1.30)

   This graph is called a *slope field*. If you follow the flow of the line segments in the slope field, you can see the general shape of the family of solutions to the differential equation

$$\frac{dy}{dx} = x.$$

   If you start at any initial point in the plane and move so that the line segments are tangent to your path, you will trace out one of the solutions.

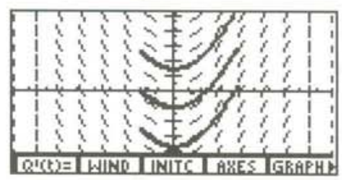

Figure 1.30

The differential equation you solved in the previous examples was easy to solve with analytic methods. In the next two examples, you will graph the solutions to differential equations that would be difficult to solve using paper and pencil.

Example 5: Normal Probability Density Functions

Graph the solution to the differential equation

$$\frac{dy}{dx} = e^{-x^2}$$

with initial conditions $y = {}^-1$ when $x = {}^-3$. Use a [⁻3,3] x [⁻1.5,1.5] viewing window.

Solution

1. Press [GRAPH] [F1] to access the differential equation editor (**Q'(t) =** screen), then press [2nd] [e^x] [(-)] [F1] [x^2] to enter the differential equation. (Figure 1.31)

Figure 1.31

2. Press [2nd] [M2] to access the window editor. Set **tMin/tMax** equal to ⁻3 and 3. These are the same values you will enter for **xMin/xMax**. (For all of the problems in this chapter, it's a good idea to begin by setting **tMin/tMax** to the same values as **xMin/xMax** on the window editor, unless a different value of **tMin** is given in the problem.)

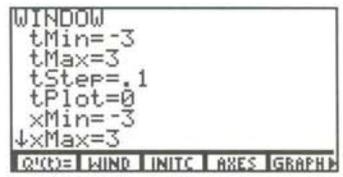

Figure 1.32

3. Change **tStep** to 0.1. This will cause the calculator to connect points every tenth starting at $x = {}^-3$.

4. Set the window to [⁻3, 3] x [⁻1.5, 1.5] by entering those values into **xMin/xMax** and **yMin/yMax**. Leave the values for **xScl** and **yScl** at 1. (Figures 1.32 and 1.33)

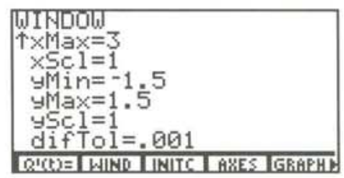

Figure 1.33

5. Press [F3] (**INITC**) and enter the initial conditions in the initial conditions editor. (Figure 1.34)

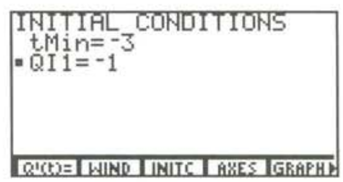

Figure 1.34

6. Check the axes by pressing [F4] (**AXES**). (Figure 1.35)

   The axes editor shows that the slope field format has been selected. The y axes will correspond to *Q1*. The size and spacing of the line segments in the slope field is determined by the value stored in **fldRes** (field resolution), which defines the number of rows displayed on the screen.

Figure 1.35

7. Now press [2nd] [M5] to see the graph of the solution to the differential equation. (Figure 1.36)

   This graph is a scaled version of *a normal probability density function*. Normal probability density functions are widely used in statistics and probability.

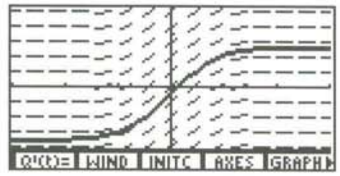

Figure 1.36

Example 6: Specifying Initial Values Interactively with the Explore Key

Graph the slope field associated with the differential equation

$$\frac{dy}{dx} = (\sin x)(\sin y) .$$

Then graph several solutions corresponding to specific initial values.

Solution

1. Press [GRAPH] [F1] (**Q'(t)=**) to enter the differential equation editor. Enter the equation using **t** for **x** and **Q1** for **y**. Press [SIN] [F1] to enter **sin t**, and then press [SIN] [F2] [1] to enter **sin Q1**.

Figure 1.37

2. Set the viewing window to [⁻6,6] x [⁻6,6], by pressing [2nd] [M2] and entering the values in the appropriate fields. (Figures 1.38 and 1.39)

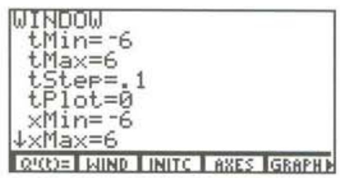

Figure 1.38

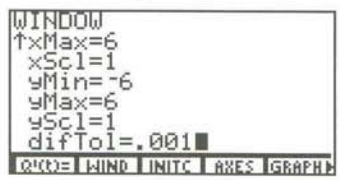

Figure 1.39

3. Since no initial conditions were specified, delete the value stored in **QI1** by pressing [F3] (**INITC**) [CLEAR]. (Figure 1.40)

   When no initial conditions are specified in the initial conditions editor, the TI-86 will graph only the slope field.

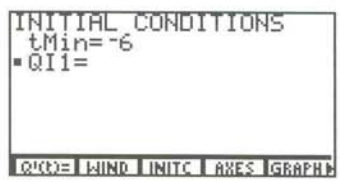

Figure 1.40

4. Check to see that the slope field format has been selected by pressing [MORE] [F1] (**FORMT**). Check the **AXES** settings by pressing [F4] (**AXES**). The value for y should be $Q1$.

5. Press [2nd] [M5] (**GRAPH**) to see the slope field. (Figure 1.41)

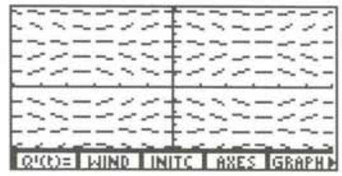

Figure 1.41

6. This slope field indicates that there are many different solutions depending on the choice of initial values. You can select the initial values (initial point) graphically with the graph explore feature (**EXPLR**). Press [MORE] to display this option above the [F5] key. (Figure 1.42)

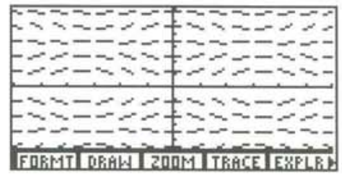

Figure 1.42

7. Now press [F5] (**EXPLR**). You should see the graph cursor blinking on and off at the origin. Use the cursor movement keys to move this cursor to the initial point shown in figure 1.43.

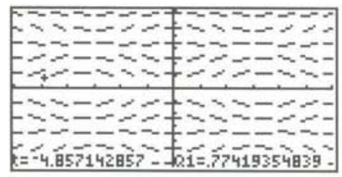

Figure 1.43

8. This points corresponds to the initial values $x = {}^-4.857142857$, $y = .77419354839$. The solution that will be drawn when you press [ENTER] will be the curve that starts at this point and is tangent to the nearby line segments.

   What do you think this curve will look like? Press [ENTER] to find out. (Figure 1.44)

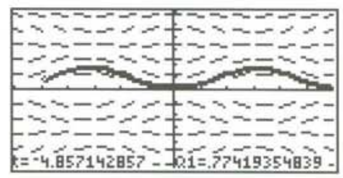

Figure 1.44

9. After the curve is drawn, the cursor reappears. You can move to another point and press [ENTER] to see the curve that corresponds to that initial point. (Figure 1.45)

   You can continue using the cursor to interactively select initial conditions as many times as you wish. When you are done, press [EXIT] to return to the GRAPH menu.

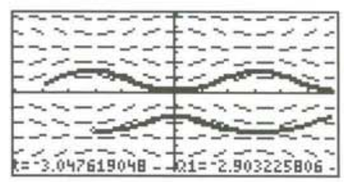

Figure 1.45

In Example 7, you will see how to graph solutions to two different differential equations simultaneously using different graphing styles.

Example 7: Graphing Simultaneous Solutions to Two Differential Equations

Simultaneously graph the solutions to the differential equations

$$\frac{dy}{dx} = y$$

with the initial point (0, .5) and

$$\frac{dy}{dx} = -y$$

with the initial point (0,10). Use a [0,3] x [0,10] viewing window.

Solution

1. Enter the first equation as $Q'1 = Q1$. You should use a different graphing style for the first equation to distinguish its graph from the second equation. You can do this with the graph style feature. After entering the differential equation in the differential equation editor, press [MORE] to access the style feature. Each time you press [F3] (**STYLE**), you will see the icon to the left of **Q'1** change shape. Each icon corresponds to a different graphing style. Select \ (thin style) shown in Figure 1.46.

Figure 1.46

2. Since you have two different equations, you will need to use a different variable for y in the second equation. Use $Q2$ for y in this equation. Press [ENTER] [(-)] [MORE] [F2] [2] to enter the second differential equation. The default style for **Q'2** should be \ (thick). Figure 1.47)

Figure 1.47

3. Enter the [0,3] x [0,10] viewing window and the initial conditions for both equations. (Figures 1.48 through 1.50)

Figure 1.48

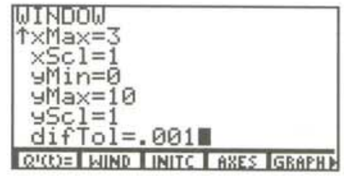

Figure 1.49

Figure 1.50

4. On the format screen ([GRAPH] [MORE] [F1]), select **FldOff**. (Figure 1.51)

5. Press [F4] to display the axes editor, and then enter **x = t** and **y = Q**. (Figure 1.52)

Figure 1.51

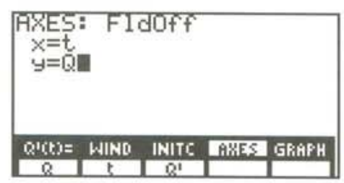

Figure 1.52

6. Graph the solutions and then press [CLEAR] to remove the menu bar from the screen. (Figure 1.53)

The thin graph is the solution to the first differential equation and the thick graph is the solution to the second equation. You can press [EXIT] to bring back the menu bar.

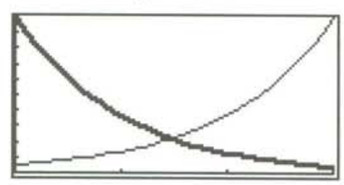

Figure 1.53

7. Use the cursor movement keys to move the graph cursor until it is positioned at the intersection of the two graphs. The x and y coordinates at the bottom of the screen approximate the simultaneous solution to the system of differential equations. (Figure 1.54)

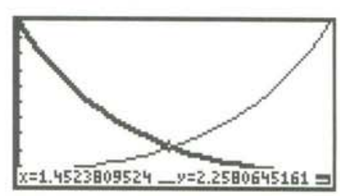

Figure 1.54

Exercises

1. Find the slope field for the differential equation

$$\frac{dy}{dx} = \cos(x)\,.$$

2. Graph the solution to the differential equation

$$\frac{dy}{dx} = \cos(x)$$

with the initial conditions $y = 0$ when $x = 0$. Use the viewing window: **xMin** = $^-2\pi$, **xMax** = 2π, **yMin** = $^-2$, **yMax** = 2 (that is, $[^-2\pi, 2\pi]$ by $[^-2,2]$). Graph the solution first with the slope field and then without.

3. Find the analytic solution to the initial value problem in Exercise 2. Use **DrawF** to overlay the analytic solution with the graph you made in Exercise 2. Remember that the **DrawF** command uses x rather than t.

4. Change the initial conditions in Exercise 2 to $y = -2$ when $x = 0$, then graph the new solution. You will need to change the viewing window to see the new graph.

5. Graph the slope field for the differential equation

$$\frac{dy}{dx} = \frac{\sin x}{x}.$$

Use the viewing window $[^-7, 7]$ by $[^-3, 3]$. What does this slope field tell you about the solution to the differential equation? Sketch a possible solution.

6. Graph the solution to the differential equation in Exercise 5 with the initial conditions $x = {^-7}$, $y = {^-1}$.

7. Redo Exercise 6, but turn off the slope field before graphing the solution.

8. Graph the slope field for the differential equation

$$\frac{dy}{dx} = x + y$$

in the $[^-10,10]$ x $[^-10,10]$ viewing window.

9. Use a list to add the specific solutions to the slope field in Exercise 8 corresponding the initial values

$x = {^-1}, y = {^-1}$

$x = {^-1}; y = 0$

$x = {^-1}, y = 1$

10. Graph the slope field corresponding to the differential equation

$$\frac{dy}{dx} = x^2 - y^2$$

in the $[^-3, 3]$ x $[^-2, 2]$ viewing window.

11. Use the **EXPLR** (explore) feature in the GRAPH menu to graph four specific solutions to the differential equation in Exercise 10. Each solution should start in a different quadrant.

12. Simultaneously graph the solutions to the differential equations

$$\frac{dy}{dx} = \cos x \quad \text{with initial point } (0,0)$$

$$\frac{dy}{dx} = \sin x \quad \text{with initial point } (0,1).$$

Use the thin graphing style for the first differential equation and the thick style for the second equation.