What's My Locus?

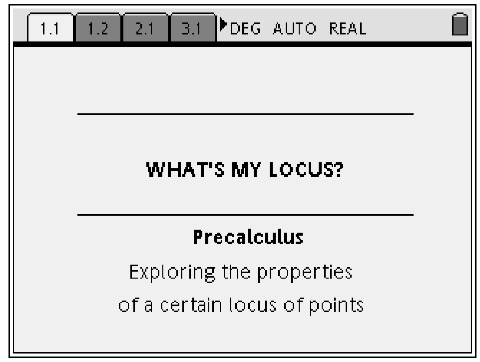
ID: 8255

Name ______________________

Class ______________________

In this activity, you will:

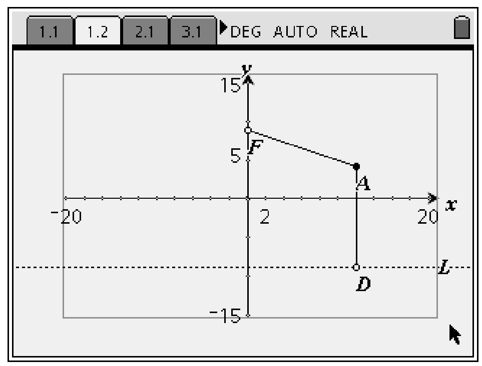
- *derive the formula for the locus of points equidistant from a fixed point and a fixed line*
- *investigate another geometric property of this locus*



Open the file *PreCalcAct2_WhatsMyLocus_EN.tns* on your handheld and work by yourself or with a partner to complete the activity. Use this document as a reference and to record your answers.

Problem 1 – Locus of points equidistant from a fixed point and a fixed line

The graph on page 1.2 shows a point *F* on the *y*-axis, a point *D* on a line *L* parallel to the *x*-axis that is the same distance below the *x*-axis as point *F* is above it, and a point *A* directly above point *D*.



Grab and drag point *F* along the *y*-axis. Observe that the relationship among point *F*, line *L*, and the origin is preserved—even if *F* is moved below the *x*-axis.

Use the **Length** tool from the Measurement menu to measure the lengths of $\overline{FA}$ and $\overline{DA}$. Move the measurements *FA* and *DA* to an open area of the screen, such as the top right corner. Then complete the following exercises.

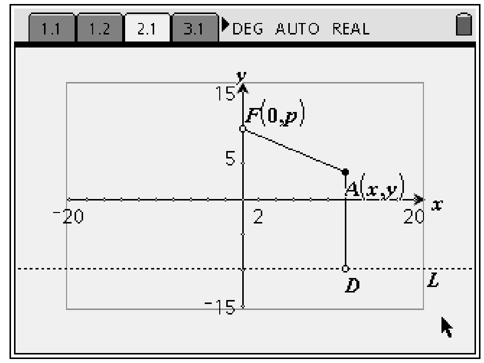
1. Grab and drag point *D* along line *L*. As you do so, focus your attention on the length measurements. What can you conclude about *FA* and *DA*?

2. Once again, grab and drag point *D*, but this time, observe the movement of point *A*. This is the locus of points that are equidistant from the point *F* and the line *L*. What curve seems to be traced out by point *A*?

3. Select the **Locus** tool from the Constructions menu to check your conjecture. After selecting the tool, click first on point *A*, followed by point *D*, and the locus should be displayed. Now drag *F* and observe how the curve changes. Does this locus match your conjecture?

Problem 2 – Derive a formula for the locus of point *A*

The locus of the point *A* is characterized by the relation *FA* = *DA*—the distance from the point *F* to *A* and the vertical distance from line *L* to *A* are equal. Your task is now to derive a *formula* for the locus and verify that it is indeed a parabola.



The diagram on page 2.1 is static—that is, the points *F* and *D* are locked in place. Points *F* and *A* are labeled on the graph along with their coordinates: (0, *p*) and (*x*, *y*), respectively.

1. Based on the coordinates shown in the diagram and the relationship between points *A*, *F* and *D*, what are the coordinates of point *D*?

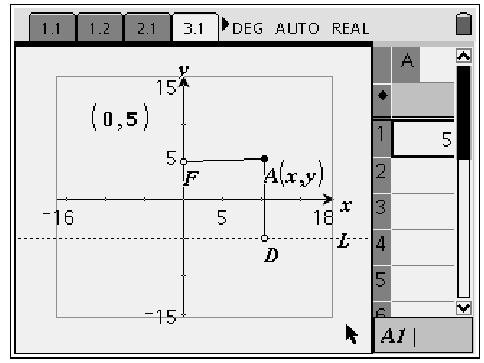
2. Using the distance formula, write expressions in terms of *x*, *y*, and *p* that represent the lengths *FA* and *DA*. Expand the expressions under the radicals.

3. Since *FA* = *DA*, set the two expressions you wrote above equal to each other and solve for *y*.

If you have correctly worked through these three exercises, you have an equation for the parabola with **vertex** at (0, 0), **focus** at *F*(0, *p*) and **directrix** $y = -p$. The distance from the vertex to the focus, $|p|$, is called the **focal length**. The vertical line that passes through the vertex, $x = 0$, is called the **axis of symmetry**.

Problem 3 – Test your understanding

You can use the derived formula found in Problem 2 to find the coordinates of the focus of a parabola with vertex at (0, 0) if *p* is unknown.



Identify the focus for each of the parabolas below.

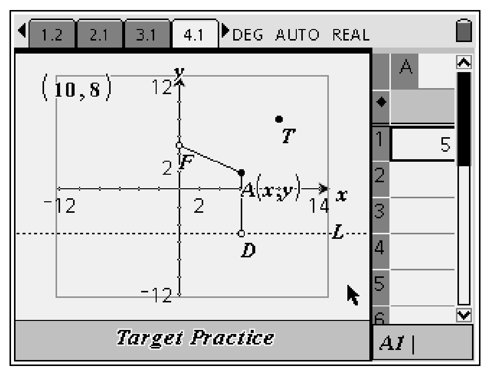
1. $y = \dfrac{x^2}{12}$ Focus: $F(0, \_\_\_)$

2. $y = -\dfrac{x^2}{36}$ Focus: $F(0, \_\_\_)$

Check your answers by using the spreadsheet and graph on page 3.1. First, enter the value for *p* into cell A1 of the spreadsheet. (Press ctrl + tab to move between the applications.) Then define **f1**(*x*) on the graph to be the known equation for the parabola. (Press ctrl + G to hide/unhide the Entry Line.) If point *A* stays on the parabola as you drag point *D*, then the value you found for *p* is correct.

Problem 4 – Target practice

You can also use that formula to find the coordinates of the focus of a parabola with vertex at (0, 0) that passes through a certain point.



On page 4.1, you will find the same set-up that is used throughout this activity, with an additional point, *T*. The coordinates of *T* are displayed in the top left corner. (Notice that *T* is restricted to integer lattice points.)

Drag *T* around the coordinate plane to a certain point (*not* the origin) and then find the focus of the parabola with vertex (0, 0) that "hits" point *T*.

$T(\_\_\_, \_\_\_)$ Focus: $F(0, \_\_\_)$

To check your answer, enter the value of *p* into cell A1 of the spreadsheet and use the **Locus** tool to display the parabola. If it passes through the point *T*, your focus is correct!

Generalize: *What if the vertex is not the origin?*

1. Given a parabola with focal length $|p|$, vertex (h, k), and axis of symmetry $x = h$, identify the following (It may be helpful to draw a diagram).

a. coordinates of the focus $F(\_\_\_\_\_, \_\_\_\_\_)$

b. equation of the directrix $y =$ _________

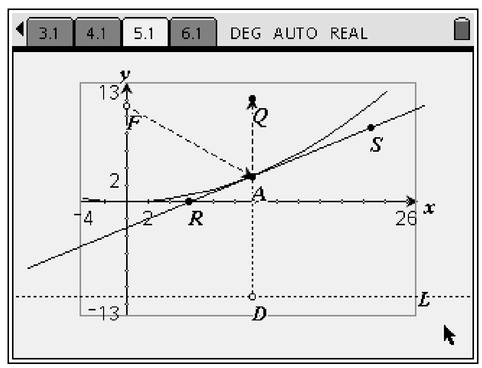
c. equation of the parabola $y =$ _________

2. Find the vertex, focus, and directrix of the parabola with equation $y = \frac{x^2}{16} - \frac{x}{2} + 6$.

Show your work.

Problem 5 – The reflection property

Page 5.1 shows a parabola and its tangent line at the point A. The line through points D, A, and Q is parallel to the y-axis. As before, the position of point A is controlled by dragging point D.



Use the **Angle** tool from the Measurement menu to display the measures of $\angle FAR$ and $\angle SAQ$. Move the measurements to an open area of the screen and complete the following exercises.

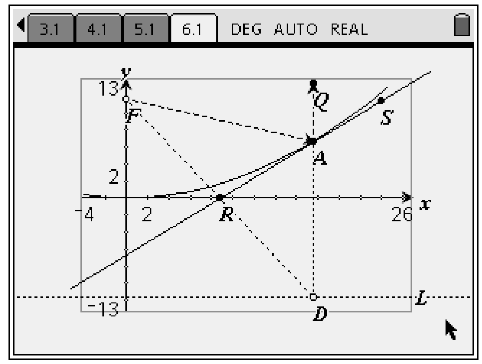
1. Grab and drag point D, causing A and its tangent line to move along the parabola. As you do so, focus your attention on the angle measurements. What can you conclude about $\angle FAR$ and $\angle SAQ$? Check your conjecture by dragging F and then dragging point D again.

2. A fact from physics tells us that the angle a light ray makes when it hits a reflective surface is always equal to the angle it makes as it reflects off of the surface, or more simply: *the angle of incidence is equal to the angle of reflection*. Suppose a light source is located at point F and the parabola is a reflective surface. What can be said collectively about all of the light rays emanating from the focus to point A, with regards to the axis of symmetry?

3. Describe two practical applications of this property, one with the focus as a transmitter and one with the focus as a receiver.

Problem 6 – A proof of the reflection property[1]

Page 6.1 displays the same diagram as page 5.1, with an additional segment ($\overline{FD}$) visible. Complete the exercises below to prove the reflection property that you explored in Problem 5.



Prove: $m\angle FAR = m\angle SAQ$

1. From Problem 2, you know that *FA* = *DA*. What does this say about $\triangle FAD$?

2. *<u>For non-calculus students:</u>* Use the **Angle** tool from the Measurement menu to display $m\angle DRA$. Does the angle change as you grab and drag *F* or *D*? What can you conclude about $\overline{FD}$ and the tangent line at point *A*?

<u>For calculus students:</u> For *A* = (*a*, *f*(*a*)), compute the slopes of $\overline{FD}$ and the tangent line at *A*. Your answer will be in terms of *a* and *p*. What can you conclude about $\overline{FD}$ and the tangent line at point *A*?

3. From Exercises 1 and 2, what can you conclude about the relationship between $\overline{AR}$ and $\triangle FAD$?

4. From Exercise 3, what can you conclude about $m\angle FAR$ and $m\angle DAR$?

5. Segments $\overline{DQ}$ and $\overline{SR}$ intersect at point *A* to form vertical angles. What is the relationship between $m\angle DAR$ and $m\angle SAQ$?

6. Conclude the proof using the transitive property of angle congruence and the results of Exercises 4 and 5.

[1] Williams, Robert C., *A Proof of the Reflective Property of the Parabola*, American Mathematical Monthly, (1987) 667-668.

