Chapter 3

Number and Operation Sense: Rational Numbers

In this chapter

This chapter summarizes NCTM's ***Principles and Standards for School Mathematics*** *for number and operation sense and applies those principles in some activity overviews. You will learn how to apply these standards while working with rational numbers in the form of fractions, decimals, and percents.*

Overview of number and operation sense: rational numbers

The whole numbers—0, 1, 2, 3, 4, 5, and so on—can be used to answer the question, "How many?" Rational numbers can be used for answering the question, "How much?"

Rational numbers that take the form of fractions, decimals, ratios, and percents mathematically represent many of the situations encountered in life. Some examples are: measurement involving parts of units (1/2 inch or 0.1 meter), relationships between two quantities (1.3 grams per liter), and measures of concentration (a 20% solution).

Goals for students

According to the *Principles and Standards for School Mathematics*, all students in grades 3-5 should:

- develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers;
- use models, benchmarks, and equivalent forms to judge the size of fractions;
- recognize and generate equivalent forms of commonly used fractions, decimals, and percents . . .
- develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students' experience;
- use visual models, benchmarks, and equivalent forms to add and subtract commonly used fractions and decimals . . .
  (NCTM, 2000, p. 148)

Young children learn the concept of fractions as they try to share a whole object fairly with others. When there are two people sharing, the whole object is divided into two equal parts, and each person gets one of the two equal parts. Eventually, children learn the mathematical name for these parts, one half, and that it takes two whole numbers together in a symbol, 1/2, to describe the result of this sharing. Similarly, if there are three people sharing, the whole must be divided into three equal parts, each person getting one of the three equal parts, or one third, written 1/3. Extensive exploration with concrete models of parts of wholes and sets provides the basis for students to develop rational number sense.

Calculators, in conjunction with physical materials, can be used by elementary students to strengthen their knowledge about rational numbers and the many different types of symbols used to represent them, as illustrated in the sample activities in the following sections.

Children in the elementary grades can also use calculators to begin to develop a sense of rational operation sense through activities that involve benchmarks and estimation, as shown in the following sections. As elementary students explore adding and subtracting fractions and decimals with the calculator, connecting the operations to physical representations and looking for patterns in their results, they develop operation sense. Then, as students learn paper-and-pencil and mental algorithms for computing, students can use the calculator as a device for exploring the mathematical principles represented in the algorithms. As students use the calculator for computation with rational numbers in more complex situations, the number and operation sense they have developed in their early explorations provides them with estimation skills that allow them to determine whether their computational results are reasonable or not.

Goals for teachers

According to the Conference Board of the Mathematical Sciences, in their document *The Mathematical Education of Teachers*:

> Having developed a variety of models of whole number operations, teachers are ready to consider how these ideas extend to integers and rational numbers. First they must develop an understanding of what these numbers are. For integers, this means recognizing that numbers now represent both magnitude and direction. And though most teachers know at least one interpretation of a fraction, they must learn many interpretations: as part of a whole, as an expression of division, as a point on the number line, as a rate or as an operator. Teachers may have learned rules for comparing fractions, but now, equipped with a choice of representations, they can develop flexibility in determining relative size.
>
> Another area to be explored is the extension of place-value notation form whole numbers to finite decimals. Teachers must come to see that any real number can be approximated arbitrarily closely by a finite decimal, and they must recognize that the rules for calculating with decimals are essentially the same as those for whole numbers. Explorations of decimals lend themselves to work with calculators particularly well (CBMS, 19).

Sample activity

Activity: Patterns in Rational Numbers

This activity uses the calculator to collect a set of data in order to identify patterns that illustrate important rational number concepts: the relationship between rational numbers and division, and equivalent fraction and decimal forms for rational numbers. Choose a divisor to use with a variety of dividends. Explore the different forms in which the quotient can be displayed on the calculator. Make a chart like the one below (but use as many rows as necessary). Use the chart to record each of the quotients in (1) integer quotient with remainder form, (2) non-simplified fraction form, (3) simplified fraction form, (4) mixed number form, and (5) decimal form. Organize the information in the chart by ordering the dividends and look for patterns. Write down three patterns that you see in your chart to share and discuss.

Patterns in Rational Numbers

Dividend	Divisor	Integer quotient with remainder	Fraction: non-simplified	Fraction: simplified	Mixed number	Decimal

Answer questions 1 - 10 as a *learner* of mathematics.

1. Look for a pattern in your tables or among tables, make a conjecture based on that pattern, and then use the calculator to test the conjecture with several other dividends of various sizes. After you have tried several dividends with a given divisor, try the same set of dividends with another divisor.

2. For which divisors do you find the same types of patterns?

3. Look at the divisor you are using; what kind of remainders do you expect?

4. What does the remainder represent?

5. How do the fraction quotients relate to the integer quotients with remainders?

6. What happens when the divisor is prime?

7. If you change the divisor but keep the same dividends, how do the remainders and fractions change?

8. Under what conditions of dividend and divisor (numerator and denominator) does the decimal representation terminate?

9. Under what conditions of dividend and divisor (numerator and denominator) does the decimal representation repeat?

10. Can you find some pair of whole numbers for divisor and dividend that will produce a decimal that does not terminate or repeat?

Answer questions 11 - 14 as a *teacher* of mathematics.

11. What did you use the calculator for in this activity?

12. How did the use of the calculator contribute to finding patterns, making conjectures, and testing them?

13. How would the experience have been different if you had not used the calculator? How might you have thought differently?

14. What ideas about rational numbers did you learn that you might not have noticed without the calculator?

Answers and comments

Below is an example of the beginning of a chart for the divisor **6**.

Patterns in Rational Numbers

Dividend	Divisor	Integer quotient with remainder	Fraction: non-simplified	Fraction: simplified	Mixed number	Decimal
1	6	0 R1	1/6	1/6	0 1/6	0.1666…
2	6	0 R2	2/6	1/3	0 1/3	0.333…
3	6	0 R3	3/6	1/2	0 1/2	0.5
4	6	0 R4	4/6	2/3	0 2/3	0.666…

Question **1**

Some patterns that students might observe include:

- The remainder matches the numerator of the fraction when it is written as a mixed number in simplest form.
- In a mixed number quotient, before being simplified, the whole number part matches the integer part of the integer quotient with remainder. For example, 8/6 = 1 2/6, or 1 remainder 2.
- For terminating decimals, if you add two of the decimal representations together, and add the equivalent fraction representations together, the sums are equivalent expressions.

  For example: 1/5 + 1/5 = 2/5, 0.2 + 0.2 = 0.4, and 2/5 = 0.4.
- When the divisor is 7, there is an interesting pattern in the repeating parts of the decimals.
- For every divisor, *n*, every *n*th dividend gives a whole number quotient with remainder 0.

Question **2**

Students should be led to compare the patterns in the divisors of 2, 4, and 8; 2, 5 and 10; 3 and 6. Why are the patterns for a divisor of 7 so different from all the rest of the divisors?

Question **3**

Students should come to the conclusion that each divisor has a finite set of remainders, from 0 to one less than the divisor itself.

Question **4**

Students should discuss the meaning represented by an integer quotient and remainder, for example 5 ÷ 3 = 1 R2, where 1 represents how many objects end up in each group of 3 (or how many groups of 3 can be made) and the 2 representing how many whole objects left over because there is not enough to form another group. Another mathematical representation of this quotient and remainder is 3(1) + 2 = 5.

Question **5**

Students should compare the above integer quotient and remainder form to the rational number quotient in fraction form, 5/3 or 1 2/3, indicating how many whole and partial objects can be placed in 3 equally-sized groups (or how many whole and partial groups of 3 can be formed). Mathematically, this relationship can be represented as 3(5/3) = 5, with no extras to be added to the product to recreate the dividend.

Question **6**

Prime number divisors other than 2 and 5 produce repeating decimals, unless they divide evenly into the dividend.

Question **7**

Many observations can be made. For example, increasing the divisor by 1 increases the cycle of whole number quotients by 1.

Question **8**

For a rational number to be represented by a terminating decimal, the prime factorization of the denominator of its simplest fraction form must contain only 2s and 5s.

Question **9**

Any rational number fraction that does not satisfy the description in 8 will be represented by a repeating decimal.

Question **10**

Here students will start trying large primes, for example 17 or 23, for divisors. They quickly have to resort to paper and pencil because the calculator's display runs out of room. By carrying out the long division algorithm with paper and pencil, they begin to see what happens: the decimal starts repeating when one element of the finite set of remainders reoccurs. At this point, they have all the ideas they need to define a rational number as a quotient of two integers, as well as mathematical properties to support that each rational number will be represented by a terminating or repeating decimal. As students become convinced that it is impossible for the quotient of two integers where the divisor is not equal to 0 to produce a non-terminating, non-repeating decimal, discussions can begin about the existence of numbers outside of the set of rational numbers, for example, irrational numbers such as pi, the square root of 2, and 0.010010001 . . .

Questions **11 – 14**

In these questions, it is important to notice that, once the connections between dividend and numerator and divisor and denominator are established, it is probably not necessary to use the calculator to produce the fraction forms of the rational numbers. However, it is also important to notice that the calculator provides *all* students with the ability to convert a large number of fractions to decimals in a short amount of time in order to collect enough data to exhibit patterns in the remainders, the fraction quotients, and the decimal quotients—patterns from which students can generalize important characteristics of rational numbers.

Using calculators to assess rational number and operation sense

Using the calculator in the assessment of understanding rational numbers and their equivalent symbolic forms, along with understanding of rational number operations, requires asking for more than just a computational result. Since operations with fractions have much less of an emphasis in the elementary grades than does the development of understanding of the numbers themselves, the role of the calculator in the assessment items at the elementary level for rational number understanding should be to collect enough data to be able to look for patterns and make conclusions about numbers or results of operations.

Assessment questions

In both of the following assessment questions, students use their understanding of rational numbers and operations to create examples to test the statements and then use the calculator to test these examples in order to support or disprove the statements.

1. True or false: If you add the same positive number to both the numerator and denominator of a fraction, the new fraction is greater than the original fraction. Justify your answer.

2. True or false: The product of a rational number and 3/4 is always less than 3/4. Use your calculator either to create a set of examples that exhibit why this statement might be true or find an example that shows that the statement is not true.

Answers and comments

Question **1**

Students should compare pairs of rational numbers such as 1/2 and 2/3, 1/3 and 2/4, 2/3 and 3/4, and so on. Students can use the calculator to make these comparisons by changing each of the fractions to decimal form or subtracting one fraction from the other to see if the difference is positive or negative. Although examples will not prove that the statement is true, students can create a large set of examples and organize them in a way that generates a pattern from which they can produce a convincing argument about why the statement appears to be true.

Question **2**

Students can create a large set of examples of p/q times 3/4, use the calculator to perform the calculation, and observe the results. Although it probably will not take many trials to find a counterexample, students may want to find a set of counterexamples upon which to build their reasoning process, for example, the statement is not true for $p/q \geq 1$.

Activity overviews for K-6: rational numbers and operations

The following list contains brief descriptions of elementary school activities that use the calculator as a recording or exploring device for developing understanding of rational numbers and rational number operations. The activities can be found on the CD that accompanies this text.

Patterns in Counting with Decimals (Schielack, Jane F. and Chancellor, Dinah. *Uncovering Mathematics with Manipulatives and Calculators, Levels 2 & 3*, pp. 21-23. Texas Instruments, 1995.)

Students use the repeating function feature, CONS feature, or OP feature of the Math Explorer™, Explorer Plus™, TI-15, or TI-73 to connect concrete and symbolic representations of tenths and hundredths in both fraction and decimal form and to recognize patterns in the number symbols.

Fraction to Decimal Division Table (Math Teacher Education Short Course - Elementary: Number Sense, pp. 35-37)

Students use the Math Explorer™, Explorer Plus™, TI-15, or TI-73 to build connections between equivalent fraction and decimal representations and identify patterns in these representations.

Fraction Forms (*TI-15: A Guide for Teachers*, pp. 6-10. Texas Instruments, 2000.)

Students use the TI-15 to compare the results of using division to create fractions under the TI-15's different mode settings for fraction display and make generalizations from the patterns they observe.

Packing the Right Snack (Math Teacher Education Short Course – Middle School: Number Relations, pp. 16-17)

Students use the Math Explorer™, Explorer Plus™, TI-15, or TI-73 to organize and represent data about the fat content in snacks in fraction, decimal, and percent forms. Note that this activity compares calories of fat to total calories in one serving, whereas on many food labels, fat content is based on grams of fat compared to total grams in one serving. Students could use the calculator to find both and compare the percentages.

Gemini Candy (Nast, Melissa, ed. *Discovering Mathematics with the TI-73: Activities for Grades 5 and 6*, pp. 61-68. Texas Instruments, 1998.)

Students use the fraction, decimal, listing, and graphing capabilities of the TI-73 to organize and represent data they have collected in order to draw conclusions about the data.

Comparing Decimals—Wider Isn't Always Larger! (Browning, C. A., and St. John, D. *Walking the Line: Activities for the TI-73 Number Line Application*, pp. 45-50. Texas Instruments, 1999.)

Students use the TI-73 and the Number Line Application to compare and order decimals.

Summing the Parts (Williams, S. E., and Bright, G. W. *Investigating Mathematics with Calculators in the Middle Grades: Activities with the Math Explorer™ and Explore Plus*™, pp. 51-56. Texas Instruments, 1998.)

Students use the Math Explorer™, Explorer Plus™, TI-15, or TI-73 to identify and extend number patterns that involve addition of fractions.

Names for One-Half (Schielack, Jane F. and Chancellor, Dinah. *Uncovering Mathematics with Manipulatives and Calculators, Levels 2 & 3*, pp.18-20. Texas Instruments, 1995.)

Students use the Math Explorer™, TI-15, or TI-73 and their understanding of integers, fractions, decimals, and operations to find mathematical expressions that equal 1/2.

Multiply Your Expectations (Williams, S. E., and Bright, G. W. *Investigating Mathematics with Calculators in the Middle Grades: Activities with the Math Explorer™ and Explore Plus*™, pp. 57-64. Texas Instruments, 1998.)

Students use the Math Explorer™, Explorer Plus™, TI-15, or TI-73 to investigate patterns that arise from multiplying fractions and decimals.