Quartic Inflection

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Abstract: This activity applications of derivatives. It introduces students to an interesting property of fourth degree polynomials and a method of proving that property using the TI-89 scripts. They then use the symbolic capacity of their calculator to generalize upon specific results.

NCTM Principles and Standards:
Algebra standards

a) Understand patterns, relations, and functions
b) generalize patterns using explicitly defined and recursively defined functions;
c) analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
d) use symbolic algebra to represent and explain mathematical relationships;
e) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
f) draw reasonable conclusions about a situation being modeled.

Problem Solving Standard build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof Standard

a) recognize reasoning and proof as fundamental aspects of mathematics;
b) make and investigate mathematical conjectures;
c) develop and evaluate mathematical arguments and proofs;
d) select and use various types of reasoning and methods of proof.

Key topic: Applications of derivatives. Inflection points. Third derivative. Scripts, formal proofs.

Degree of Difficulty: advanced

Needed Materials: TI-89 calculator

Situation: Quartic polynomials have many interesting properties. In this activity we'll use calculus to investigate one of them with the aid of the TI-89 calculator. Many quartic polynomials have two inflection points. Consider a line containing the two inflection points Where else does this line cross the quartic? What is the midpoint of the line segment between the two inflection points?

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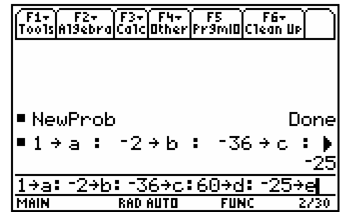
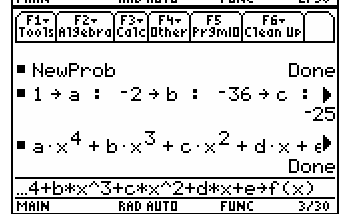
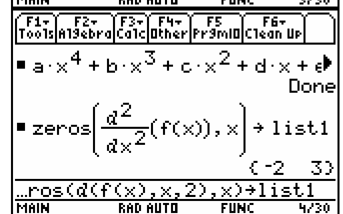
Choose arbitrary values for the coefficients of a quartic polynomial and store the result in f(x).

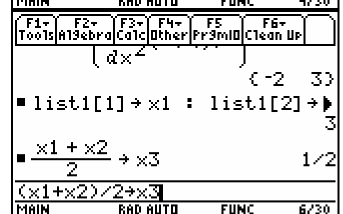




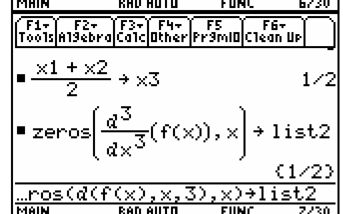
Find the zeroes of the second derivative and store the result in list. If there are no real zeros, go back and change the coefficients of the quartic:



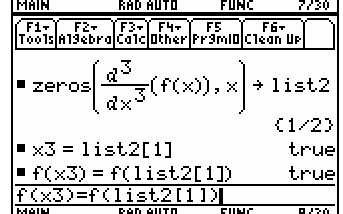
Store these zeros as x1 and x2 and store their average in x3:



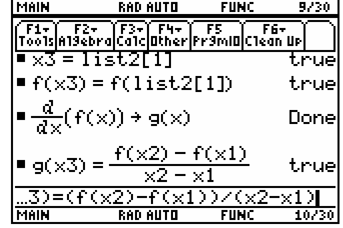
Now compute the zeros of the third derivative of f(x) and store that in list2:



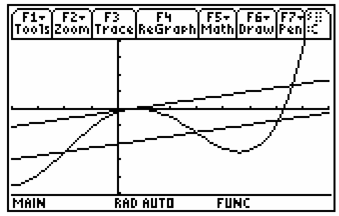
Now compare x3 and list2[1] and f(x3) with f(list2[1])



Finally, let g(x) = $\dfrac{df(x)}{dx}$ and compare g(x3) with $\dfrac{f(x2)-f(x1)}{x2-x1}$.



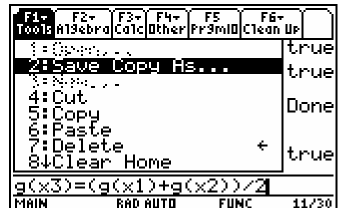
What have we shown? We've shown for this quartic that the x-coordinate midpoint of the line segment connecting the inflection points of a quartic is the x-coordinate of the point where the third derivative, $\dfrac{d^3y}{dx^3}=0$. We've also shown that the tangent line at the x-coordinate of the point where the third derivative, $\frac{d^3y}{dx^3} = 0$ is



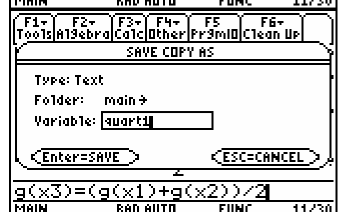
parallel to the line between the two inflection points:

Is this property always true? You could scroll back and change the coefficients of the cubic and execute the steps again, but we'll take a different tack. We'll turn what we've written into a script which can be followed for any cubic:

Press F1 and choose "Save Copy As"

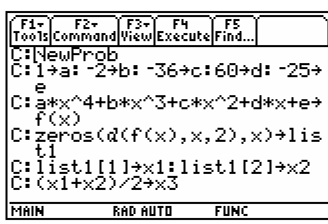


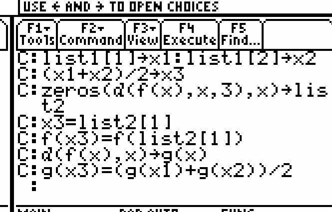
I named this script "quart1"



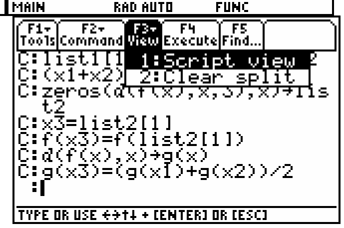
From the application menu APPS choose 8:Text Editor:

and open the script named quart1:





This script contains the commands that were originally typed in. We can play this script for other choices of a, b, c, d, and e: First press F3 to change the view to A: Script View



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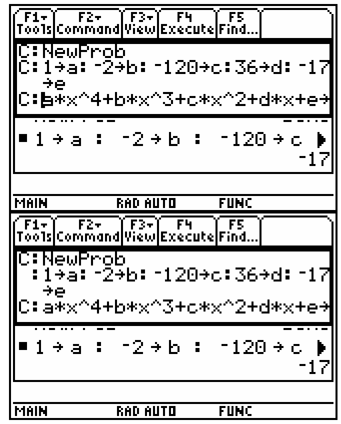
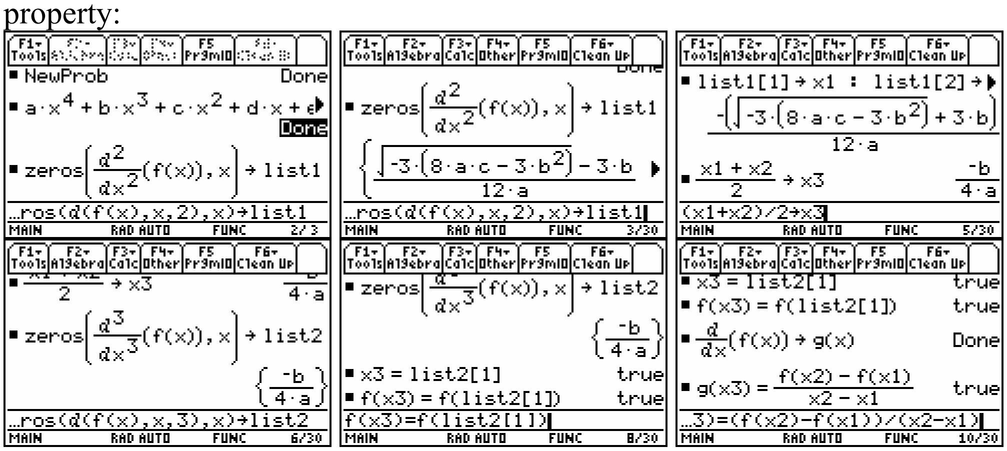
Pressing [F4] executes each line. Change the values of a, b, c, d, and e and continue pressing [F4] to see that the property is true with these choices.



Is it always true? Go back to the line "C: 1→a: -2→b: -120→c: 36→d: -17→e" and choose 4:Clear Command from the [F2] Command menu which will erase the C: in the line C: 1→a: -2→b: -120→c: 36→d: -17→e

Now run the script again and observe that the calculator creates a proof of this property:



Scripts can be very useful in proving properties. Here we have shown for any fourth degree polynomial that the x-coordinate midpoint of the line segment connecting the inflection points of the polynomial is the x-coordinate of the point where the third derivative, $\frac{d^3y}{dx^3} = 0$. We've also shown that the tangent line at the x-coordinate of the point where the third derivative, $\frac{d^3y}{dx^3} = 0$ is parallel to the line between the two inflection points. Note that the point where the third derivative, $\frac{d^3y}{dx^3} = 0$ is -b/(4a). This is very similar in form to the formula for the x-coordinate of the vertex of a parabola in standard form! Can you extend this result?

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