Quartic Inflection

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Abstract: This activity applications of derivatives. It introduces students to an interesting property of fourth degree polynomials and a method of proving that property using the TI-89 scripts. They then use the symbolic capacity of their calculator to generalize upon specific results.

NCTM Principles and Standards:

Algebra standards

- a) Understand patterns, relations, and functions
- b) generalize patterns using explicitly defined and recursively defined functions;
- c) analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
- d) use symbolic algebra to represent and explain mathematical relationships;
- e) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
- f) draw reasonable conclusions about a situation being modeled.

Problem Solving Standard build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof Standard

- a) recognize reasoning and proof as fundamental aspects of mathematics;
- b) make and investigate mathematical conjectures;
- c) develop and evaluate mathematical arguments and proofs;
- d) select and use various types of reasoning and methods of proof.

Key topic: Applications of derivatives. Inflection points. Third derivative. Scripts, formal proofs.

Degree of Difficulty: advanced

Needed Materials: TI-89 calculator

Situation: Quartic polynomials have many interesting properties. In this activity we'll use calculus to investigate one of them with the aid of the TI-89 calculator. Many quartic polynomials have two inflection points. Consider a line containing the two inflection points Where else does this line cross the quartic? What is the midpoint of the line segment between the two inflection points?

	■1→a: -2→b: -36→c: ▶ -25
	1+a: -2+b: -36+c:60+d: -25+e
	F1+ F2+ F3+ F4+ F5 F6+ Too1sA19ebra[Ca1c Other Pr9mI0[Clean Up]
	NewProb Done
	■1→a: -2→b: -36→c:) -25
	■ a·x ⁴ + b·x ³ + c·x ² + d·x + e
	4+b*x^3+c*x^2+d*x+e+f(x)
Find the zeroes of the second derivative and store the result in list. If	Finite Fight
there are no real zeros, go back and change the coefficients of the	■ a·x ⁴ + b·x ³ + c·x ² + d·x + e
quartic:	= zeros $\left[\frac{a^2}{\sqrt{2}}(f(x)), x\right] $ + list1
	(ax-) (-2 3)
	…ros(d(f(x),x,2),x)→list1 MANN RADAUTO FUNC 4/30
Store these zeros as x1 and x2 and store their average in x3:	F1+ F2+ F3+ F4+ F5 ToolsA19ebraCa1cOtherPr9miOClean UP
	(-2 3) ■list1[1]→×1 : list1[2]→
	$= \frac{\times 1 + \times 2}{2} \rightarrow \times 3 \qquad 1/2$
	(x1+x2)/2+x3 MAIN RAD AUTO FUNC 5/30
Now compute the zeros of the third derivative of $f(x)$ and store that in	F1+ F2+ F3+ F4+ F5 F6+ Too1sA19ebraCa1cOtherPr9mIOClean Up
list2:	$= \frac{\times 1 + \times 2}{2} \rightarrow \times 3 \qquad 1/2$
	= zeros $\left[\frac{a^3}{\sqrt{3}}(f(x)), x\right] \rightarrow 1$ ist2
	(a×°) (1/2)
	<u>…ros(d(f(x),x,3),x)→list2</u> MAIN RADAUTO FUNC 7/30
Now compare x3 and list2[1] and f(x3) with f(list2[1])	F1+ F2+ F3+ F4+ F5 F6+ ToolsA19ebraCalcOtherPr9mlOClean Up
	• zeros $\left(\frac{d^3}{dx^3}(f(x)), x\right) $ + list2
	(ux-) (1/2)
	• $x3 = 1ist2[1]$ true • $f(x3) = f(1ist2[1])$ true
	f(x3)=f(11st2[1])
$df(\mathbf{r}) = f(\mathbf{r}^2) - f(\mathbf{r}^1)$	
Finally, let $g(x) = \frac{df(x)}{d}$ and compare $g(x3)$ with $\frac{f(x2) - f(x1)}{2}$.	■ x3 = list2[1] true
dx $x2-x1$	f(x3) = f(1ist2[1]) true
	$= \frac{1}{d \times} (f(x)) \neq g(x) \qquad \text{Done}$
	$ g(x3) = \frac{f'(x2) - f'(x1)}{x2 - x1} $ true
	3)=(f(x2)-f(x1))/(x2-x1) MAIN RAD AUTO FUNC 10/30
What have we shown? We've shown for this quartic that the x-co	ordinate
midpoint of the line segment connecting the inflection points of a quartic	is the x-
coordinate of the point where the third derivative, $\frac{d^3y}{d^3} = 0$. We've also s	hown that the
dx^{-}	

Choose arbitrary values for the coefficients of a quartic polynomial and store the result in f(x).

F1+ F2+ F3+ F4+ F5 F6+ ToolsAl9ebraCalcOtherPr9mIOClean Up

Done ::) ÷e 2/3

NewProb

tangent line at the x-coordinate of the point where the third derivative, $\frac{d^3y}{dx^3} = 0$ is



parallel to the line between the two inflection points:

Is this property always true? You could scroll back and change the coefficients of the cubic and execute the steps again, but we'll take a different tack. We'll turn what we've written into a script which can be followed for any cubic:

Press F1 and choose "Save Copy As"



TYPE OR USE ++++ CENTERI OR CE

Pressing F4 executes each line. Change the values of a, b, c, d, and e and continue pressing F4 to see that the property is true with these choices.

Is it always true? Go back to the line "C: $1 \rightarrow a: -2 \rightarrow b: -120 \rightarrow c: 36 \rightarrow d: -17 \rightarrow e$ " and choose 4:Clear Command from the F2 Command menu which will erase the C: in the line C: $1 \rightarrow a: -2 \rightarrow b: -120 \rightarrow c: 36 \rightarrow d: -17 \rightarrow e$

F1+ F2+ F3+ F4 F5 ToolsCommandViewExecuteFind
C:NewProb C:1+a:-2+b:-120+c:36+d:-17
, C : ⊨*x^4+b*x^3+c*x^2+d*x+e→
■1→a:-2→b:-120→c) -17
MAIN BAD AUTO FUNC
Tools Command View Execute Find
tóðis(commandvíðu)[zxðdute find] C:NewProb : 1+a: -2+b: -120+c: 36+d: -17 +e C:a*x^4+b*x^3+c*x^2+d*x+e+
tóðiscommánndvíéw Execute Find C:NewProb :1+a: -2+b: -120+c:36+d: -17 de C:a*x^4+b*x^3+c*x^2+d*x+e+ •1+a: -2+b: -120+c↓ -17

Now run the script again and observe that the calculator creates a proof of this property:

F1+ SC- CAVE F5 SC Tools State And State Pr9mID Class C	F1+ F2+ F3+ F4+ F5 F6+ ToolsAlgebraCalcOtherPrgmIDClean Up	F1+ F2+ F3+ F4+ F5 F6+ ToolsAl9ebraCalcOtherPr9mIDClean Up
NewProb Done	(.2)	<pre>■list1[1] → ×1 : list1[2] → ▶</pre>
■a·x ⁴ +b·x ³ +c·x ² +d·x <u>+</u> €	■ zeros $\frac{q^2}{4r^2}(f(x)), x \rightarrow 1$ ist1	-[3.(8.a.c - 3.b ²] + 3.b]
Done	(<u>ax-</u>)	12:a
• zeros $\frac{d^2}{2}(f(x)), x \rightarrow 1$ ist1	{ -3·(8·a·c - 3·b ²) - 3·b ▶	
(d×2)	12·a	- 2 7×3 4·a
ros(d(f(x),x,2),x)→list1	ros(d(f(x),x,2),x)→list1	(x1+x2)/2+x3
MAIN RADAUTO FUNC 2/3	MAIN RADIAUTO FUNC 3/30	MAIN RADIAUTO FUNC 5/30
F1+ F2+ F3+ F4+ F5 F6+ ToolsA19ebraCalcOtherPr9mIOClean Up	F1+ F2+ F3+ F4+ F5 ToolsAlgebraCalcOtherPrgmIDClean UP	F1+ F2+ F3+ F4+ F5 ToolsAl9ebraCalcOtherPr9mIOClean UP
$ \xrightarrow{1}{2} \rightarrow \times 3 \qquad \xrightarrow{1}{4} \xrightarrow{2}{3} $	• zeros $\frac{\alpha}{2}$ (f(x)), x \rightarrow list2	•x3=list2[1] true
2 7 1	[(a× ³))]	<pre>f(x3) = f(list2[1]) true</pre>
■ zeros $\left \frac{d^3}{dx^3} (f(x)), x \right \rightarrow 1ist2$	$\left\{\frac{-b}{4 \cdot a}\right\}$	$= \frac{d}{d\times}(f(x)) \neq g(x) \qquad \text{Done}$
	•x3=list2[1] true	$f(x^2) = f(x^2) = f(x^1)$
<u>∖4·a</u> ∫	<pre>f(x3) = f(list2[1]) true</pre>	$= g(x_3) = \frac{1}{x_2 - x_1}$ true
ros(d(f(x),x,3),x)+list2	f(x3)=f(list2[1])	3)=(f(x2)-f(x1))/(x2-x1)
MAIN RADIAUTO FUNC 6/30	MAIN RAD AUTO FUNC 8/30	MAIN RAD AUTO FUNC 10/30

Scripts can be very useful in proving properties. Here we have shown for any fourth degree polynomial that the x-coordinate midpoint of the line segment connecting the inflection points of the polynomial is the x-coordinate of the point where the third

derivative,
$$\frac{d^3y}{dx^3} = 0$$
. We've also shown that the tangent line at the x-coordinate of the

point where the third derivative, $\frac{d^3y}{dx^3} = 0$ is parallel to the line between the two

inflection points. Note that the point where the third derivative, $\frac{d^3y}{dx^3} = 0$ is -b/(4a). This

is very similar in form to the formula for the x-coordinate of the vertex of a parabola in standard form! Can you extend this result?