NUMB3RS Activity: A Group of Symmetries
Episode: "Waste Not"

Topic: Symmetries of an Equilateral Triangle **Grade Level:** 10 - 12
Objective: Explore and visualize the reflection and rotational symmetries of an equilateral triangle and relate them to the properties of a group.
Materials: Paper, pencil, transparencies, and transparency pens if available.
Time: 20 - 25 minutes

Introduction

Charlie prepares a class on Kac-Moody (pronounced "Kots-Moody") Algebras and he and the new division chair Dr. Finch write a Kac-Moody equation together. While these algebras are beyond the scope of most high school classes, they are an extension of groups of symmetries, which should be accessible. This activity takes the simple group of symmetries of an equilateral triangle and relates the geometry to some of the axioms high school students commonly study.

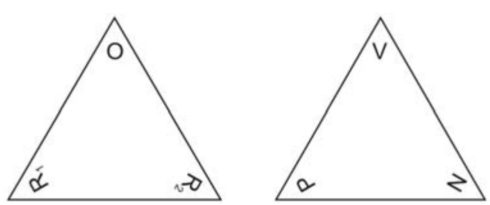
This activity is best done with students working in pairs or small teams, since a team has a better chance of catching any errors. In addition, testing the associative property in the extensions is faster and easier when the work is shared.

Discuss with Students

Review the definition of congruence for geometric figures. Point out that for an equilateral triangle, the reflection over any altitude or a clockwise rotation of 120° or 240° about the circumcenter results in a figure that is congruent to the original. Using a triangle drawn on a transparency is an excellent way to demonstrate this. Otherwise, students can cut an equilateral triangle from a piece of paper and mark each vertex as described in the activity.

Help students see that what starts as a table showing one rigid motion (like the reflection above) followed by another becomes a table of an operation that exhibits properties from algebra. In particular, this activity works with identity and inverse. It also illustrates that the commutative property does not always hold.

In the activity, students will label the triangle as shown below:



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These labels indicate the vertex that will appear at the top front after a particular transformation. For instance, after the **N** transformation, the vertex labeled N will appear at the top of the front side. This labeling is helpful to students as they complete successive transformations, because it clearly shows how a single transformation can produce the same result. For instance, for **V** followed by $\mathbf{R_1}$, the label N will appear at the top of the front side. This indicates that **V** * $\mathbf{R_1}$ = **N** (where * means "followed by").

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When your students finish, point out that in mathematics a "group" consists of a set and an operation that is closed under the operation and includes an identity and inverse for each element in the set (as well as the associative property, which is discussed in the "Extensions"). Ask them to come up with examples of groups other than real or rational numbers (e.g., the even integers under addition).

This activity is a good opportunity to discuss that while high school students see a big separation between algebra and geometry, the distinction fades as one advances in mathematics.

Student Page Answers:

1.

*	O	V	P	N	R_1	R_2
O	O	V	P	N	R_1	R_2
V	V	O	R_1	R_2	P	N
P	P	R_2	O	R_1	N	V
N	N	R_1	R_2	O	V	P
R_1	R_1	N	V	P	R_2	O
R_2	R_2	P	N	V	O	R_1

2. O *is the identity.* **3.** *Yes;* **O, V, P**, *and* **N** *are their own inverses and* **R_1** *and* **R_2** *are inverses of each other.* **4.** *It is* ***not*** *commutative*. *One counterexample is* **R_2*N** = **V** *but* **N*R_2** = **P**. *Student counterexamples may vary.* **5.** *Any of the three letters can be placed at the top of the front of the triangle. Once that is done, there are 2 letters left to place in the lower left. That leaves only one for the lower right. From the fundamental counting principle, if there are 3 ways to do one thing, 2 ways to do the next and 1 way to do the third, then there are (3)(2)(1) = 6 ways to do all of them. Thus there are 6 ways to arrange the letters (permutations) as shown on the Student Page.*

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Name: ________________________________ Date: _______________

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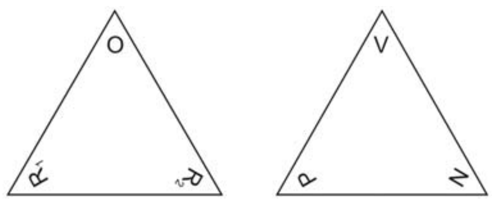
Charlie prepares a class on Kac-Moody (pronounced "Kots-Moody") algebras. He and the new division chair Dr. Finch write a Kac-Moody equation together. While these algebras are beyond the scope of high school mathematics, they are extensions of groups of symmetries. This activity explores the symmetry group of an equilateral triangle.

Consider the set of all reflections and rotations that leave an equilateral triangle congruent to the original (possibly with the vertices labeled differently). An equilateral triangle can be reflected along any of its three altitudes or rotated clockwise about the circumcenter (the intersection point of the three altitudes) by 120° or 240°, and fit on itself. (In a trivial case, it can also be rotated 0°, which leaves it as it was originally.)

At your teacher's direction, draw on a transparency or cut an equilateral triangle from a piece of paper, and label the vertices as shown below. Note that P is "behind" R_2 and N is "behind" R_1.



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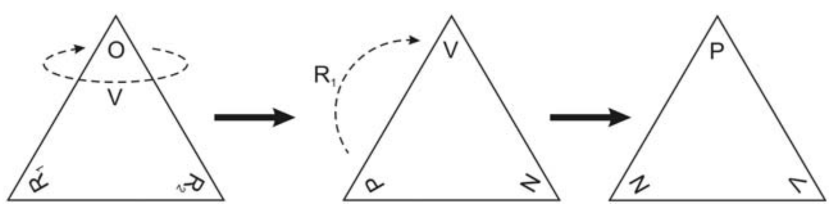
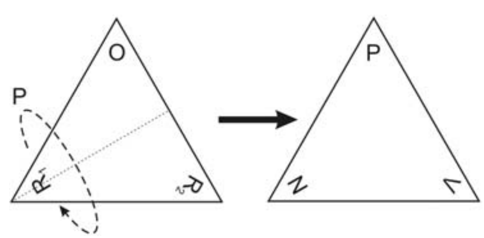
Let $\mathbf{R_1}$ be a clockwise rotation of 120° about the circumcenter and let $\mathbf{R_2}$ be a clockwise rotation of 240° about the circumcenter. Use **O** for the figure in its original position.

Let **V** be the reflection across the vertical axis, let **P** be the reflection across the altitude from the lower-left corner to the middle of the right side, and let **N** be the altitude from the lower-right corner to the middle of the left side.

The transformation **V** followed by $\mathbf{R_1}$ means to do **V** to the original triangle first and then do $\mathbf{R_1}$ to the result, as demonstrated below. (The **V** in the first figure is on the back.)



These two transformations produce the same result as a single **P** reflection. That is, **V** followed by $\mathbf{R_1}$ results in **P**, as demonstrated below.



Using ***** to represent the operation "followed by," this can be written as $\mathbf{V * R_1 = P}$.

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1. Complete the table below, showing the results of performing each operation listed down the left followed by the ones listed across the top. Sample results are provided.

*	O	V	P	N	R_1	R_2
O	O			N		
V			R_1			N
P			O			
N						P
R_1					R_2	
R_2		P				

2. The identity property of an operation states that there is an element of the set that leaves every element unchanged under the operation. From the table above, does the operation ***** have an identity? If so, what is it? If not, give a counterexample.

3. The inverse property states that every element of the set has an element (possibly itself) such that when both operations are performed, the result is the identity. Does each element in this set have an inverse under the operation ***** ? If so, list the inverse of each of the six moves in the set. If not, give a counterexample.

4. If the order does not matter for an operation on two elements, then the operation is said to be commutative (for instance, with multiplication $x \cdot y = y \cdot x$). Using the table, is the operation ***** commutative? If so, explain how the table shows this. If not, give a counterexample.

5. Show why there are no more than 6 different symmetries of an equilateral triangle.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Introduction

This activity shows how geometric properties can become the foundation for an algebraic structure – in this case a group.

For the Student

1. This activity shows some of the properties of the symmetric group on equilateral triangles. A group is a mathematical structure with four properties (identity, inverse, closure, and associativity). The activity illustrates the identity and inverse properties.

- Closure states that for every x and y in the set, the result $x * y$ is also in the set. Use the table to verify this.
- The associative property says that for all x, y, and z in the set, $x * (y * z) = (x * y) * z$. For even a small set (as in this activity), this can take some time to prove. How many cases must be examined? Team up with a few other people and each do a part to verify that this operation is associative and, thus all the group properties are satisfied.

2. The actual class Charlie prepares is on Kac-Moody algebras, which are an extension of Weyl groups. These groups include the symmetries of hexagons and squares. Use a line of reasoning similar to that used for the triangle to develop the symmetric group for the square. The square has four axes of symmetry and rotations of 90°, 180°, and 270° (as well as the 0° identity). Determine how many elements are in the table, name them appropriately, and then develop the correct table.

Additional Resources

- While these examples are in two dimensions, this concept is not limited to the plane. Start with a cube and consider all of its possible symmetries. For an interactive demonstration on a square, see:
  http://www.math.csusb.edu/notes/advanced/algebra/d4/d4.html

- For more about the symmetries of a square and a cube, see:
  http://www.maths.uwa.edu.au/~schultz/3P5.2000/3P5.2,3SquareCube.html

- For a complete pair of lessons on these symmetries from Illuminations, see:
  http://illuminations.nctm.org/LessonDetail.aspx?ID=U157

Related Topic

One of the applications of group theory is Rubik's cube. For an explanation of this relationship, along with the four properties of a group, as well as a table for the solution to Rubik's Cube, see: **http://members.tripod.com/~dogschool/cubegroups.html**

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