

## Population and Limits

by

**Mary Ann Connors**

Department of Mathematics  
Westfield State College  
Westfield, MA 01086

### Textbook Correlation: Key Topic

- Limits

### NCTM Principles and Standards:

- Process Standard
  - Representation
  - Connections
  - Problem Solving

Consider a model of a population that lives in an unfavorable habitat. Assume 20% of the population dies in a given generation and 100 more immigrates to the habitat at the end of each generation. Describe the population as time passes. Will this population die out eventually? Given the initial population is 200, the model is described by the recursion formula:

$$p_0 = 200$$

$$p_n = 0.8p_{n-1} + 100$$

where

$n = 0, 1, 2, 3, \dots$  years and

$p_n$  is the number of insects after  $n$  years.

### Numerical Analysis

Enter the equation and initial condition in sequence mode on your TI-89 or TI-92 Plus. Investigate the table and time series plot:

F1	F2	F3	F4	F5	F6	F7
Tools	Zoom	Edit	✓	All	Style	Axes...
*PLOTS						
✓ u1 = .8 · u1(n - 1) + 100						
u1 = 200						
u2 =						
u3 =						
u4 =						
u2(n) =						
MAIN    RAD    AUTO    SEQ						

F1	F2	F3	F4	F5	F6	F7
Tools	Setup	Table	Plot	Seq	Stat	Draw
n	u1					
0.	200.					
1.	260.					
2.	308.					
3.	346.4					
4.	377.12					
n=0.						
MAIN    RAD    AUTO    SEQ						

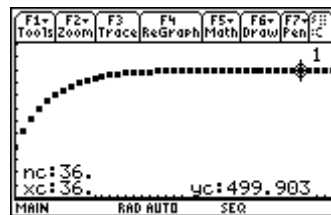
F1+ Tools	F2+ Setup	F3+ Data	F4+ Matrix	F5+ Calc	F6+ 1/d	F7+ Stat
n		u1				
50.		499.996				
51.		499.997				
52.		499.997				
53.		499.998				
54.		499.998				
n=54.						
MAIN		RAD AUTO		SEQ		

From observing the iterations on the table, **it appears that** after 30 generations the population is leveling off and approaching a limit of 500.

## Graphical Analysis

Graphing the model.

F1+ Tools	F2+ Zoom
nmin=0.	
nmax=40.	
plotStart=1.	
plotStep=1.	
xmin=0.	
xmax=40.	
xsc1=1.	
ymin=0.	
ymax=600.	
ycs1=50.	
MAIN	
RAD AUTO	
SEQ	



The time series plot also illustrates graphically that the population **appears** to be leveling off or approaching a limit of about 500. The horizontal axis (x scale =1) represents the number of generations (time interval) and the vertical axis (y scale = 50) represents the population (number of inhabitants). Note that when you trace, the n, x and y coordinates of the marked point on the graph are given at the bottom of the TI-89 screen.

## Finding the Limit of a Sequence

Let's look at the sequence again. This time we will enter it in the Data/Matrix Editor (**Apps, 6**). Follow the steps illustrated on the following screens:

F1+ Tools	APPLICATIONS					
	1:Home					
	2:Y= Editor					
	3:Window Editor					
	4:Graph					
	5:Table					
	1:Current Matrix Editor					
	2:Open... am Editor					
	3:New... Editor					
MAIN						
RAD AUTO						
SEQ						

NEW	
Type:	Data
Folder:	main
Variable:	pos
Row dimension:	2
Col dimension:	2
Enter=OK	
ESC=CANCEL	
MAIN	
RAD AUTO	
SEQ	

Highlight c1. Access the **MATH** menu (2<sup>nd</sup>, 5). Select **3:List** and then **1:seq(**.

F1+ Tools	F2+ Plot Setup	F3+ Cell	F4+ Reader	F5+ Calc	F6+ 1/d	F7+ Stat
DATA						
	c1	c2	c3			
1						
2						
3						
4						
c1=						
MAIN		RAD AUTO		SEQ		

F1+ Tools	F2+ Plot Setup	F3+ Cell	F4+ Reader	F5+ Calc	F6+ 1/d	F7+ Stat
MATH						
	1:Number					
	2:Angle					
	3:List					
	4:Matrix					
	5:Complex					
	6:Statistics					
	7:Probability					
	8:Test					
c1=						
MAIN		RAD AUTO		SEQ		

F1+ Tools	F2+ Plot Setup	F3+ Cell	F4+ Reader	F5+ Calc	F6+ 1/d	F7+ Stat
MATH						
	1:Number					
	1:seq(					
	2:min(					
	3:max(					
	4:SortA					
	5:SortD					
	6:sum(					
	7:cumSum(					
	8:product(					
TYPE OR USE $\leftrightarrow$ + [ENTER] OR [ESC]						

After pressing **ENTER**, complete the command by typing **x,x,0,40**).

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	c1	c2	c3			
1						
2						
3						
4						
c1=seq(x,x,0,40)						
MAIN RAD AUTO SEQ						

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	c1	c2	c3			
1	0					
2	1					
3	2					
4	3					
r1c1=3						
MAIN RAD AUTO SEQ						

Highlight c2, select **seq(** from the Math menu again. Complete the command with **u1(n),n,0,40**. Make sure that nmin is 0 in the Window editor. Otherwise you will get a sequence setup error message.

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	c1	c2	c3			
1	0					
2	1					
3	2					
4	3					
c2=						
MAIN RAD AUTO SEQ						

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	c1	c2	c3			
1	0					
2	1					
3	2					
4	3					
c2=seq(u1(n),n,0,40)						
MAIN RAD AUTO SEQ						

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	c1	c2	c3			
1	0	200.				
2	1	260.				
3	2	308.				
4	3	346.4				
r1c2=200.						
MAIN RAD AUTO SEQ						

Add another column to the table. Take the absolute value of the difference between the current population and the apparent limit, 500, (a reasonable guess). After highlighting c3, go to the MATH menu to select **abs(**.

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	c1	c2	c3			
1	0	200.				
2	1	260.				
3	2	308.				
4	3	346.4				
c3=						
MAIN RAD AUTO SEQ						

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
MATH						
To						
1	Number					
2	Angle					
3	List					
4	Matrix					
5	Complex					
6	Statistics					
7	Probability					
8	Test					
c3=						
MAIN RAD AUTO SEQ						

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
MATH						
To						
1	exact(					
2	abs(					
3	round(					
4	iPart(					
5	fPart(					
6	floor(					
7	ceiling(					
8	sign(					
c3=						
MAIN RAD AUTO SEQ						

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	c1	c2	c3			
1	0	200.				
2	1	260.				
3	2	308.				
4	3	346.4				
c3=abs(500-c2)						
MAIN RAD AUTO SEQ						

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	c1	c2	c3			
1	0	200.	300.			
2	1	260.	240.			
3	2	308.	192.			
4	3	346.4	153.6			
r1c3=300.						
MAIN RAD AUTO SEQ						

You may assign titles to the columns if you wish. Simply highlight the space above the desired column, press **ENTER** and type the title followed by **ENTER**.

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	c1	c2	c3			
1	0	200.	300.			
2	1	260.	240.			
3	2	308.	192.			
4	3	346.4	153.6			
c1,Title=						
MAIN RAD AUTO SEQ						

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	c1	c2	c3			
1	0	200.	300.			
2	1	260.	240.			
3	2	308.	192.			
4	3	346.4	153.6			
c1,Title=gen						
MAIN RAD AUTO SEQ						

F1+ Tools	F2 Plot Setup	F3 Cell	F4 Header	F5 Calc	F6+ Util	F7 Stat
DATA						
	gen	pop	1500-...			
	c1	c2	c3			
1	0	200.	300.			
2	1	260.	240.			
3	2	308.	192.			
4	3	346.4	153.6			
c1,Title="gen"						
MAIN RAD AUTO SEQ						

**Note:** An alternative way to look at the values in the tables above is to define a new function in the Y= editor,  $u_2(n)$  = the absolute value of the difference between the limit and the sequence..

For example,

F1	F2	F3	F4	F5	F6	F7
Tools	Zoom					
*PLOTS						
✓ $u_1 = .8 \cdot u_1(n-1) + 100$						
$u_{i1} = 200.$						
✓ $u_2 =  500 - (.8 \cdot u_1(n-1) + 100) $						
$u_{i2} = 300.$						
$u_3 =$						
$u_{i3} =$						
$u_2(n) = \text{abs}(500 - (.8 \cdot u_1(n-1) + 100))$						
MAIN RAD APPROX SEQ						

F1	F2	F3	F4	F5	F6	F7
Tools	Setup					
n	u1	u2				
0.	200.	300.				
1.	260.	240.				
2.	308.	192.				
3.	346.4	153.6				
4.	377.12	122.88				
n=0.						
MAIN RAD APPROX SEQ						

### Exercise 1:

When will the population be within a tolerance  $\epsilon = 100$  of 500? That is, when will  $|500 - p_n| = |p_n - 500| < 100$ ?

**Answer:**

Looking at the figures in the table, for  $n \geq 5$ ,  $|p_n - 500| < 100$ .

F1	F2	F3	F4	F5	F6	F7
Tools	Plot Setup	Cell	Header	Calc	Util	Stat
DATA	gen	pop	1500-...			
	c1	c2	c3			
5	4	377.12	122.88			
6	5	401.7	98.304			
7	6	421.36	78.643			
8	7	437.09	62.915			
r6c1=5						
MAIN RAD AUTO SEQ						

Mathematically, we represent this by saying, for  $\epsilon = 100$  there is a number  $N = 5$  such that for  $n \geq 5$ ,  $|p_n - 500| < 100$ . This means that after five or more generations the population will be within a tolerance of 100 of the determined limit.

If the tolerance = 10, the corresponding  $N$  is 16 because for  $n \geq 16$ ,  $|p_n - 500| < 10$ .

F1	F2	F3	F4	F5	F6	F7
Tools	Plot Setup	Cell	Header	Calc	Util	Stat
DATA	gen	pop	1500-...			
	c1	c2	c3			
9	8	449.67	50.332			
10	9	459.73	40.265			
11	10	467.79	32.212			
12	11	474.23	25.77			
r12c1=11						
MAIN RAD AUTO SEQ						

F1	F2	F3	F4	F5	F6	F7
Tools	Plot Setup	Cell	Header	Calc	Util	Stat
DATA	gen	pop	1500-...			
	c1	c2	c3			
13	12	479.38	20.616			
14	13	483.51	16.493			
15	14	486.81	13.194			
16	15	489.44	10.555			
r16c1=15						
MAIN RAD AUTO SEQ						

F1	F2	F3	F4	F5	F6	F7
Tools	Plot Setup	Cell	Header	Calc	Util	Stat
DATA	gen	pop	1500-...			
	c1	c2	c3			
16	15	489.44	10.555			
17	16	491.56	8.4442			
18	17	493.24	6.7554			
19	18	494.6	5.4043			
r17c1=16						
MAIN RAD AUTO SEQ						

### Exercise 2:

What is the corresponding  $N$  for  $\epsilon = 1$ ?

**Definition of limit of a sequence:**

A sequence of numbers,  $p_1, p_2, p_3, \dots$  approaches the limit  $L$   
(written  $\lim_{n \rightarrow \infty} p_n = L$ ), if for every tolerance  $\varepsilon > 0$ , there is some  
positive integer  $N$  such that for every integer  $n \geq N$ ,  $|p_n - L| < \varepsilon$ .

**Challenge:**

From your experimental evidence it appears that the population is approaching a limit of 500. How can you know for sure that this continues? Use the definition of limit of a sequence to verify that 500 is the limit of the sequence  $p_n$  as  $n$  approaches infinity.