Population and Limits

by

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Textbook Correlation: Key Topic

- Limits

NCTM Principles and Standards:

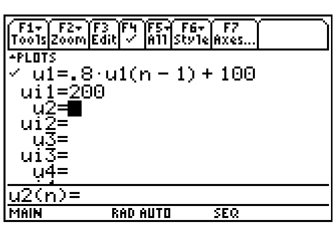
- Process Standard
    - Representation
    - Connections
    - Problem Solving

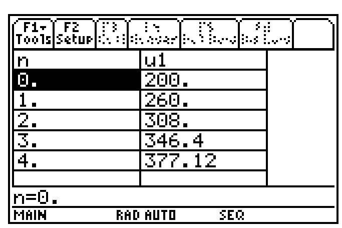
Consider a model of a population that lives in an unfavorable habitat. Assume 20% of the population dies in a given generation and 100 more immigrates to the habitat at the end of each generation. Describe the population as time passes. Will this population die out eventually? Given the initial population is 200, the model is described by the recursion formula:

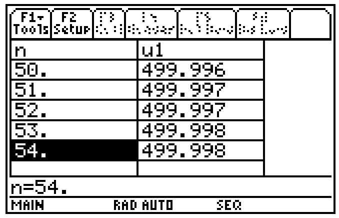
$p_0 = 200$
$p_n = 0.8p_{n-1} + 100$
where
$n = 0,1,2,3\ldots$years and
p_n is the number of insects after n years.

Numerical Analysis

Enter the equation and initial condition in sequence mode on your TI-89 or TI-92 Plus. Investigate the table and time series plot:



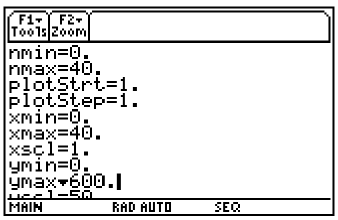


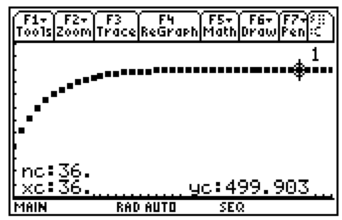


From observing the iterations on the table, **it appears that** after 30 generations the population is leveling off and approaching a limit of 500.

Graphical Analysis

Graphing the model.

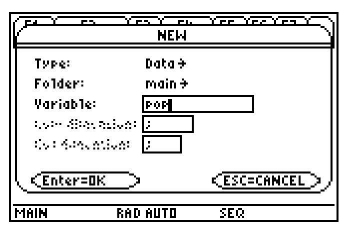




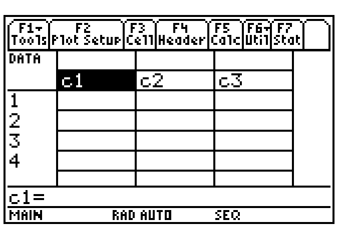
The time series plot also illustrates graphically that the population **appears** to be leveling off or approaching a limit of about 500. The horizontal axis (x scale =1) represents the number of generations (time interval) and the vertical axis (y scale = 50) represents the population (number of inhabitants) . Note that when you trace, the n, x and y coordinates of the marked point on the graph are given at the bottom of the TI-89 screen.

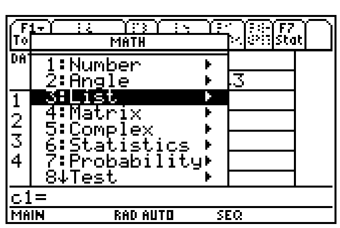
Finding the Limit of a Sequence

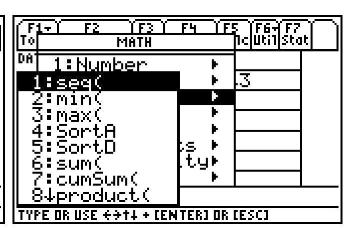
Let's look at the sequence again. This time we will enter it in the Data/Matrix Editor (**Apps**, **6**). Follow the steps illustrated on the following screens:



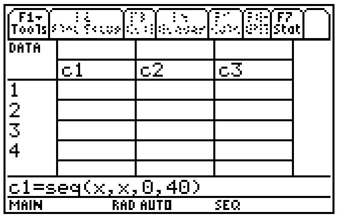
Highlight c1. Access the **MATH** menu (**2nd**, **5**). Select **3:List** and then **1:seq(**.

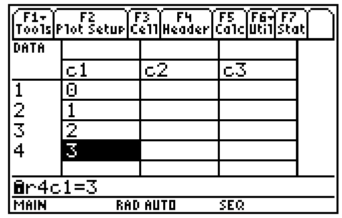




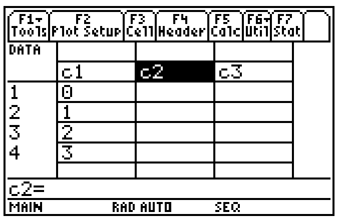


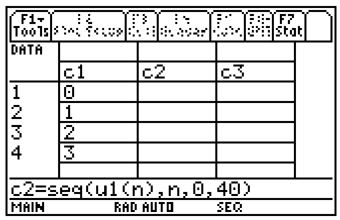
After pressing **ENTER**, complete the command by typing **x,x,0,40)**.

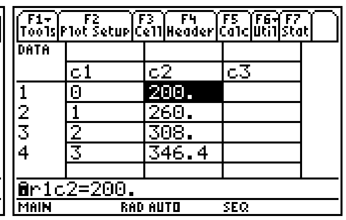




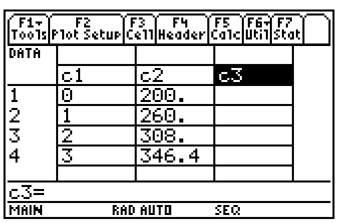
Highlight c2, select **seq(** from the Math menu again. Complete the command with **u1(n),n,0,40)**. Make sure that nmin is 0 in the Window editor. Otherwise you will get a sequence setup error message.

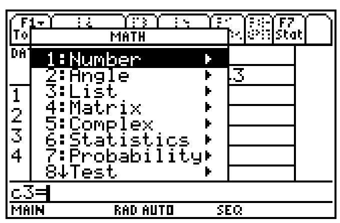


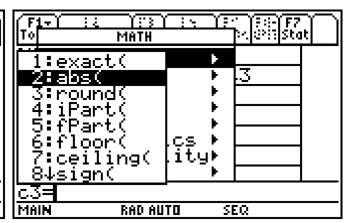


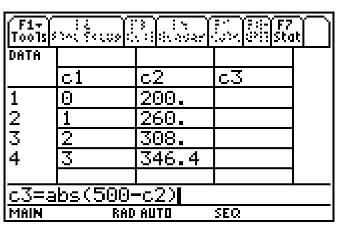


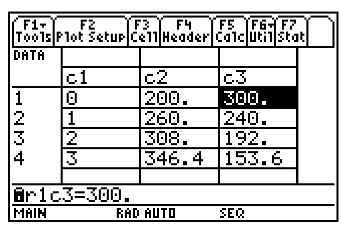
Add another column to the table. Take the absolute value of the difference between the current population and the apparent limit, 500, (a reasonable guess). After highlighting c3, go to the MATH menu to select **abs(**.



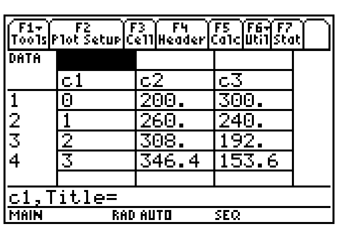


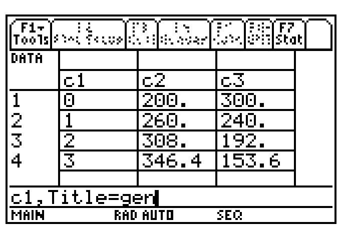


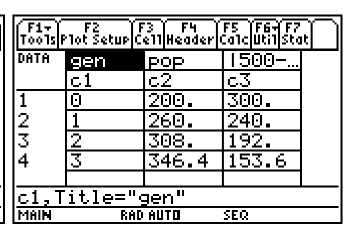




You may assign titles to the columns if you wish. Simply highlight the space above the desired column, press **ENTER** and type the title followed by **ENTER**.

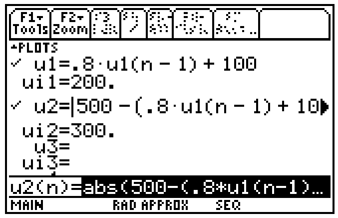


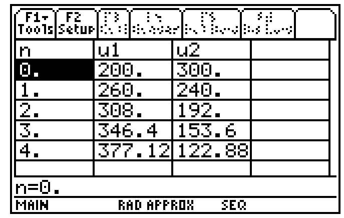




Note: An alternative way to look at the values in the tables above is to define a new function in the Y= editor, u2(n) = the absolute value of the difference between the limit and the sequence..

For example,

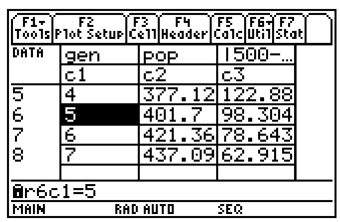




Exercise 1:

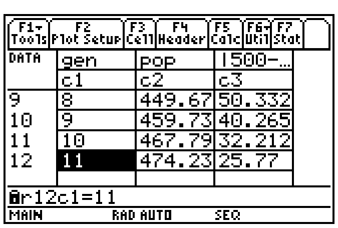
When will the population be within a tolerance $\varepsilon = 100$ of 500? That is, when will $|500 - p_n| = |p_n - 500| < 100$?

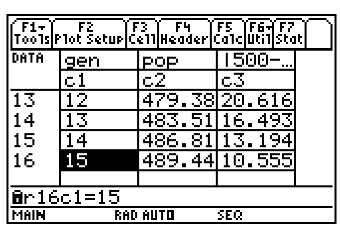
Answer:
Looking at the figures in the table, for $n \geq 5$, $|p_n - 500| < 100$.

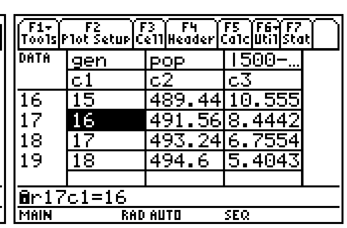


Mathematically, we represent this by saying, for $\varepsilon = 100$ there is a number N = 5 such that for $n \geq 5$, $|p_n - 500| < 100$. This means that after five or more generations the population will be within a tolerance of 100 of the determined limit.

If the tolerance = 10, the corresponding N is 16 because for $n \geq 16$, $|p_n - 500| < 10$.







Exercise 2:

What is the corresponding N for $\varepsilon = 1$?

Definition of *limit of a sequence*:

A sequence of numbers, p_1 , p_2 , p_3 , ... approaches the limit L (written $\lim_{n \to \infty} p_n = L$), if for every tolerance $\varepsilon > 0$, there is some positive integer N such that for every integer $n \geq N$, $\lvert p_n - L \rvert < \varepsilon$.

Challenge:
From your experimental evidence it appears that the population is approaching a limit of 500.How can you know for sure that this continues? Use the definition of limit of a sequence to verify that 500 is the limit of the sequence p_n as n approaches infinity.