NEWTON'S METHOD

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(Please feel free to email me questions and /or comments.)

Key Topic: Applications of Derivatives

Abstract:

In this activity we explain how and why Newton's method works. And we give an example which explains in detail how to use the TI-89 to execute Newton's method. Then the students are given an exercise which asks them to use Newton's method to approximate $\sqrt{3}$.

Prerequisite Skills:

- Ability to use the derivative to find the equation of the tangent to a curve
- Ability to graph functions using a TI-89

Degree of Difficulty: Easy

Needed Materials: TI-89

NCTM Principles and Standards:

- Content Standards – Algebra
    - Represent and analyze mathematical situations and structures using algebraic symbols
    - Use mathematical models to represent and understand quantitative relationships
    - Draw a reasonable conclusion about situation being modeled

- Content Standards - Geometry
    - Use geometric modeling to solve problems

- Process Standards
    - Representation
    - Connections
    - Problem Solving

NEWTON'S METHOD

Newton's method is an algorithm which makes use of the derivative of a function to approximate the zeros of the function. To understand how and why Newton's method works, let's first concentrate on the derivative of a function. One of the first things you learned about the derivative was that it could be used to find the slope, and thus also the equation, of the tangent to a curve at a specified point.

And one of the really nice things about the tangent to a curve is that it can, under certain circumstances, be used to approximate the curve. To see this, consider figure 1 where we have graphed a curve and its tangent at the point where $x = a$. When we "zoom in" on the point of tangency, as is shown in figure 2, we see that around the point of tangency, it becomes difficult to distinguish between the graph of the curve and the graph of the tangent. If we "zoom in" again, as we did in figure 3, it becomes even more difficult to distinguish between the curve and the tangent. So in the "window" depicted in figure 3, the points on the tangent can be used to approximate the points on the curve.

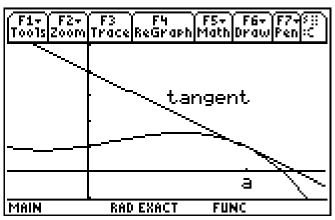


Figure 1

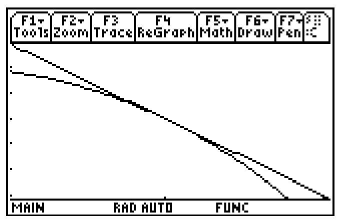


Figure 2

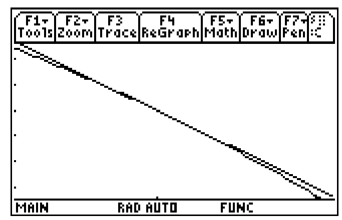
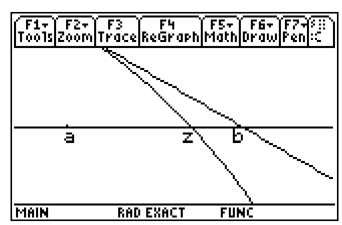


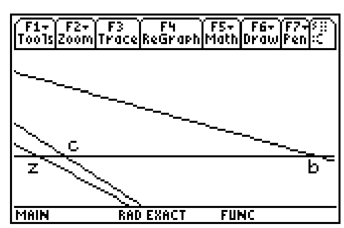
Figure 3

But can we use the points on the tangent to approximate the points on the curve which are not close to the point of tangency? In particular, can we use the zero of the tangent to approximate the zero of a function?

In figure 1, the zero of the function and the zero of the tangent look kind of close, but we know that if we "zoom in" on these zeros they will no longer look very close. This is pictured at the right where *a* is the *x*-coordinate of the point of tangency, z is the zero of the function, and b is the zero of the tangent. In the situation, the zero of the tangent doesn't give a very good approximation of the zero of the function.



But what if we moved the point of tangency closer to the zero of the function? Since b is closer to z than a is, let's create a tangent to the function at the point where $x = b$. The result is pictured at the right where c is the zero of the new tangent.



As we can see, c is a much better approximation of z than b was. And we could probably get an ever better approximation if we continued this process by looking the zero of the tangent at the point where $x = c$.

This is the basis of Newton's method. That is, to approximate a zero of a function $f(x)$:

1. Pick a value x_1 which is close to the zero of the function.
2. Find the zero x_2 of the tangent to $f(x)$ at the point $(x_1, f(x_1))$.
3. Find the zero x_3 of the tangent to $f(x)$ at the point $(x_2, f(x_2))$.
4. Continue this iterative process.

Each new x_n should be a better approximation of the zero of the function than the previous approximation x_{n-1}. But what is a formula for x_n?

Since $f'(x_{n-1})$ is the slope of the tangent to $f(x)$ at the point $(x_{n-1}, f(x_{n-1}))$, the equation of this tangent is $y - f(x_{n-1}) = f'(x_{n-1}) \cdot (x - x_{n-1})$. And it is easy to show that the zero of this tangent is $x = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$. But in Newton's method, this zero is denoted by x_n. Hence

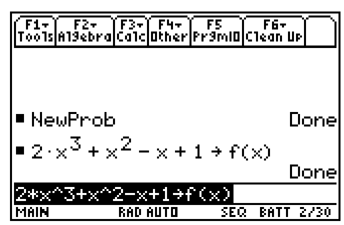
$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

EXAMPLE: Use Newton's method to approximate the zeros of $f(x) = 2x^3 + x^2 - x + 1$.

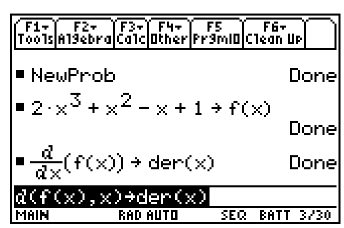
The steps below show you how to do this using a TI-89.

Whenever you start a new problem, clear memory by pressing [2nd][F6]2.

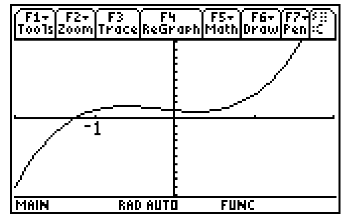
1. Define the function $f(x) = 2x^3 + x^2 - x + 1$ by using the [STO▸] key to store $2x^3 + x^2 - x + 1$ in $f(x)$.



2. Store the derivative of $f(x)$ in der(x).
[F3] [ENTER] [alpha] **f** [(] [X] [)] [,] [X] [)] [STO▸]
[2nd] [alpha] **d e r** [alpha] [(] [X] [)] [ENTER]

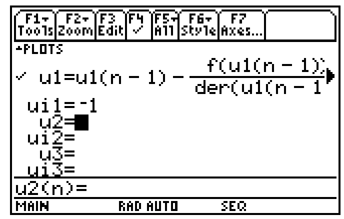


3. Graph the function by pressing [◆] [Y=] and setting **y1** equal to $f(x)$. Pick a value x_1 which is close to the zero of the function. We will take $x_1 = -1$.

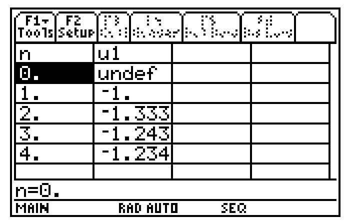


4. Put calculator in sequence mode. [MODE] ⊙ [4] [ENTER] and press [◆] [Y=]. In terms of the notation we developed above, **u1(n)** denotes x_n. Also, **u1** and **n** can be entered into the calculator using the [alpha] key.

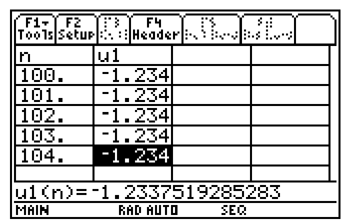
5. Set **u1** equal to Newton's Method. Set **ui1** equal to -1, our initial guess at the zero of $f(x)$.



6. Press [◆] [TABLE] to see the results.



7. Press [◆] [TblSet] and set **tblStart** to a larger number, such as 100. Press [ENTER] twice to see how Newton's Method converges on the zero. (This may take awhile.)



From this we see that the zero, accurate to two decimal places, is -1.23.

EXERCISE: Use Newton's method to approximate $\sqrt{3}$.