Implicit differentiation

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Abstract: This activity is an example of a differentiation technique. Students learn how to differentiate an equation in two variables with respect to one of them. They learn how to calculate points where the tangent line is horizontal, vertical, or oblique. They then use the graphing capacity of their calculator to display the graph of the implicit function.

NCTM Principles and Standards:
Algebra standards
a) Understand patterns, relations, and functions
b) use symbolic algebra to represent and explain mathematical relationships;
c) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.

Geometry standards:
a) Analyze characteristics and properties of two- and three-dimensional geometric shapes and mathematical about geometric relationships
b) visualize three-dimensional objects and spaces from different perspectives and analyze their cross sections;

Problem Solving Standard build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.

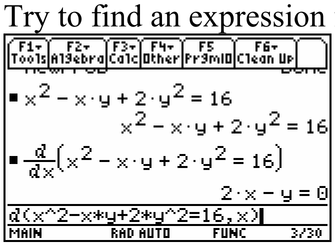
Reasoning and Proof Standard
a) recognize reasoning and proof as fundamental aspects of mathematics;
b) make and investigate mathematical conjectures;
c) develop and evaluate mathematical arguments and proofs;
d) select and use various types of reasoning and methods of proof.

Key topic: Derivatives - implicit differentiation

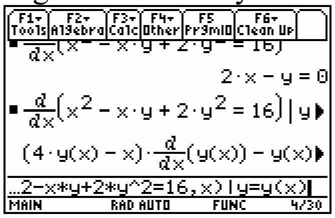
Degree of Difficulty: Moderate to advanced
Needed Materials: TI-89 calculator

Situation:
Consider the curve whose equation is $x^2 - xy + 2y^2 = 16$

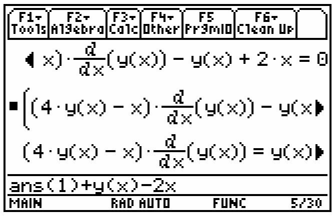
Try to find an expression for dy/dx:

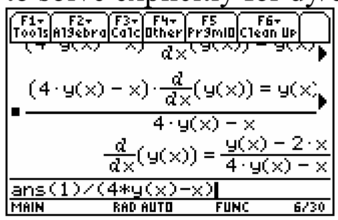


The calculator treats y as a constant rather than as a function of x. That is why the result is given incorrectly as: 2x - y = 0.

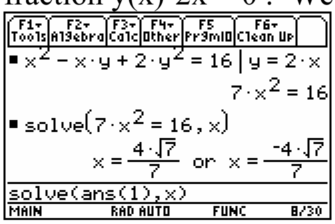


We can use the calculator to solve explicitly for dy/dx:

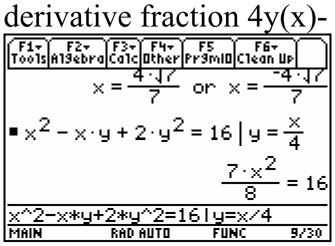


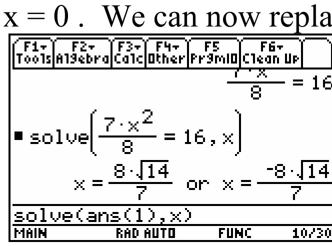


To find the x-coordinates where the tangent line is horizontal set the numerator of the fraction y(x)-2x = 0 . We can now replace y with 2x in the original equation:

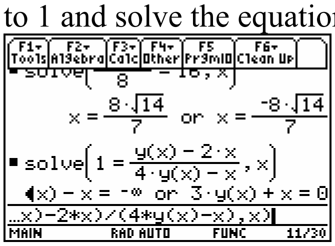


To find the x-coordinates where the tangent line is vertical set the denominator of the derivative fraction 4y(x)- x = 0 . We can now replace y with x/4 in the original equation:

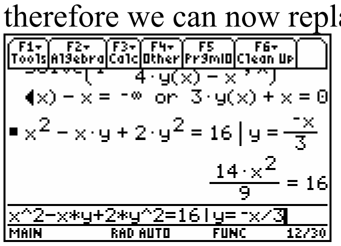


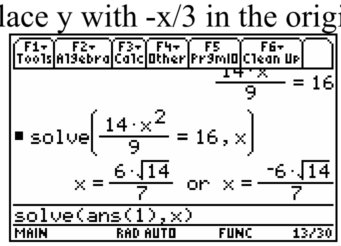


To find the x-coordinates where the tangent line has a slope of 1, set the derivative equal to 1 and solve the equation for x in terms of y:



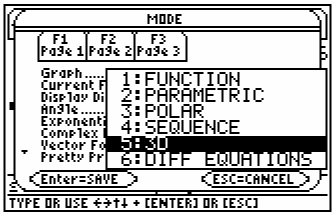
Note: The only real answer occurs when 3y(x) + x = 0 and therefore we can now replace y with -x/3 in the original equation:

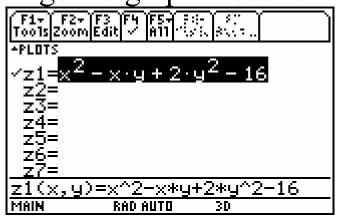


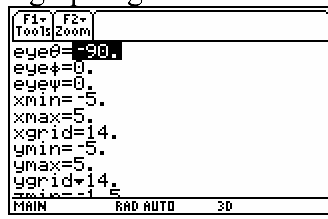


Finally, we can use the implicit plot features of the TI-89 to graph the equation. First we need to rewrite it so that one side is equal to zero: $0 = x^2 - xy + 2y^2 - 16$

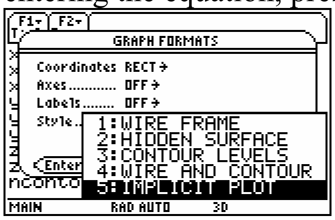
Use the Mode button to change the graph mode to 3D graphing:

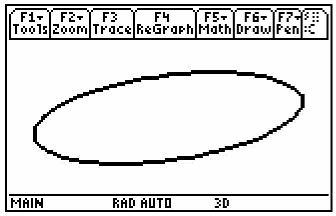






After entering the equation, press F1 and scroll down to 9:Format, then choose Implicit Plot:





Essentially the calculator is plotting a particular slice of a three dimensional solid: the slice when $z = 0$. We can use the F5:Math menu to display the position on the graph of a point where the tangent line is vertical: