The Beach Race

By Stephen Arnold

Time required
90 minutes

ACTIVITIES eXCHANGE **ID: 8320**

Activity Overview

This activity presents the following problem: A beach race begins from a point 4 kilometers out to sea from one end of a 6-kilometer beach and finishes at the opposite end of the beach. Contestants must swim to a point along the beach and then run to reach the finish line. I can swim at 4 km·h^{-1} and run at 10 km·h^{-1}. Where should I aim to land on the beach so as to minimize my total time for the race? Students will explore the solution to this problem using TI-Nspire CAS technology.

Background

This problem has a distinctly Australian flavor and has long been popular as an optimization problem involving differential calculus in senior years. Using appropriate technological tools, however, it is also accessible as a valuable algebraic modeling experience for a much wider range of students.

Concepts

- *The Pythagorean Theorem, algebraic modeling, maxima and minima using calculus*

Teacher Preparation

Some discussion of the nature of mathematical modeling and the key concepts of variable and domain would be valuable in introducing this activity. Students should expect to move from a geometric model to a numerical one; move from numbers to graphs; and finally derive an algebraic expression, which may be verified using the graph previously obtained. The purpose of this algebraic model should be made clear—it allows for a much closer investigation of the situation, and will support students in making meaningful predictions. Also, specific discussion of the relationship among distance, speed, and time may be necessary.

- *The screenshots on pages 62–64 demonstrate expected student results. Refer to the screenshots on pages 65 and 66 for a preview of the student TI-Nspire document (.tns file).*
- ***To download the student .tns file, go to education.ti.com/exchange and enter "8320" in the quick search box.***

Classroom Management

- *Some introductory **teacher-led** discussion, making explicit early interpretations of the task, is very important. Students should always be encouraged to articulate what they expect to happen before they begin using technological tools. Pen-and-paper sketches of the problem should lead quickly to the interactive geometric models capable of generating data points.*
- *It is always valuable to begin a challenging activity with students working for at least a few minutes individually (in order to foster a level of commitment to the task) and then encouraging students to work in pairs. Later, pairs can join into groups of four, in order to provide further stimulus for discussion and further support for those students who may be struggling.*

- *The student TI-Nspire solution document CalcAct13_BeachRace_Soln_EN.tns shows the expected results of working through the activity (lengths measured, expressions calculated, data collected, and answers recorded).*
- *Ideas for optional extensions are provided on page 65.*

TI-Nspire™ Applications

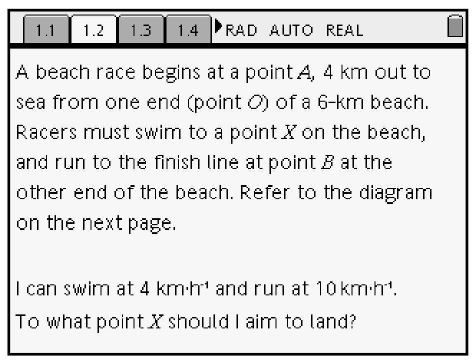
Calculator, Graphs & Geometry, Lists & Spreadsheet, Notes

Technical Prerequisites

Students should know how to

- *construct simple geometric models, using segments and variable points, and measure the lengths and distances relevant to the task (Graphs & Geometry application)*
- *substitute measured values into an appropriate formula (entered using the* ***Text*** *tool) and store these values for further study (Calculator, Graphs & Geometry, and Lists & Spreadsheet applications)*
- *store numerical values as variables in order to build an algebraic model that forms the basis of prediction (Calculator, Graphs & Geometry, and Lists & Spreadsheet applications)*

Step 1: Introductory discussion and sharing of ideas



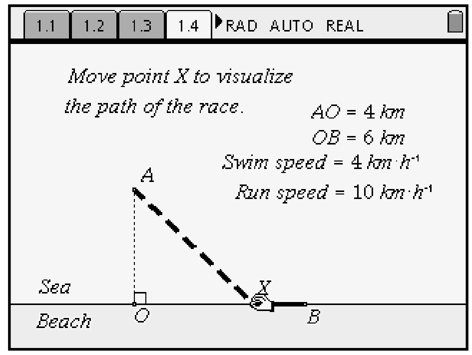
Introductory discussion could center on the relationship among distance, speed, and time, with students providing examples from their own experience. (For example, traveling 20 kilometers to school should take 20 minutes at an average speed of 60 km·h^{-1}.)

Personal attributes should also be featured in this discussion. “How fast do you walk? How fast do you think you can swim?” If time allows, use of a motion detector would support and motivate students in this, as well as building strong understanding of the factors involved in this task.

The context of the beach race may then be introduced and some time for student discussion allowed. Either individually or in pairs, students should be encouraged to sketch a pen-and-paper diagram of the problem situation as they understand it.

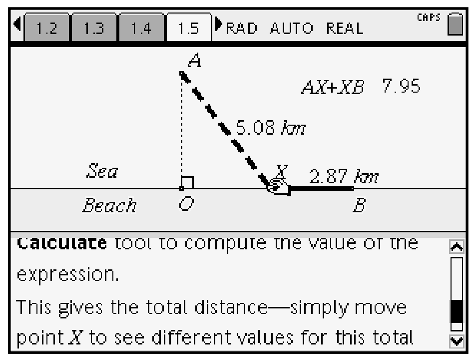
Step 2: Modeling with the *Graphs & Geometry* application

The introduction of the model in the *Graphs & Geometry* application should be seen as a natural extension of students' previous pen-and-paper modeling. Individually or in pairs, students should attempt to construct their own models of the beach race and begin to take relevant measurements. Some class discussion may be needed at regular intervals, allowing students to share understandings and techniques with others who may need additional support.

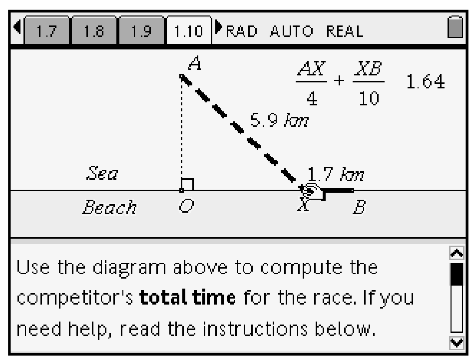


Step 3: Building a "Total Time" formula

Before finding a formula for the total time of the race, it would be helpful to begin with an exploration of the total ***distance*** of the race, using the model provided on page 1.5.



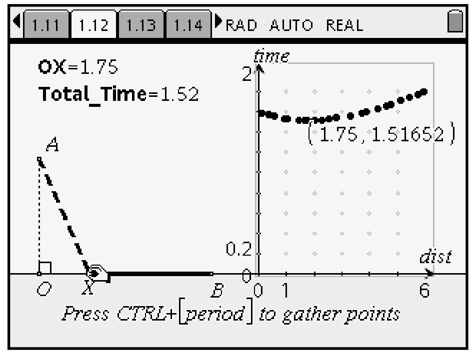
From there, you can lead in to the idea of the total time for the race. This key concept needs to be clear to all students—the total time for the race is the sum of the time needed for the swim and the time needed for the run. Students will explore finding this time via a series of questions, culminating with verification using the model on page 1.10.

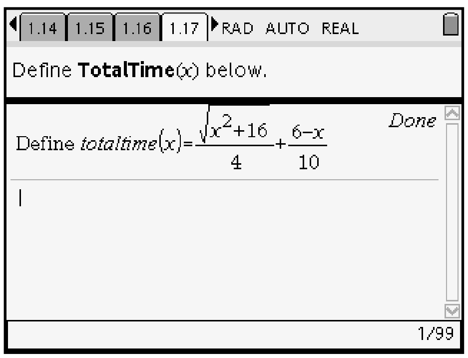


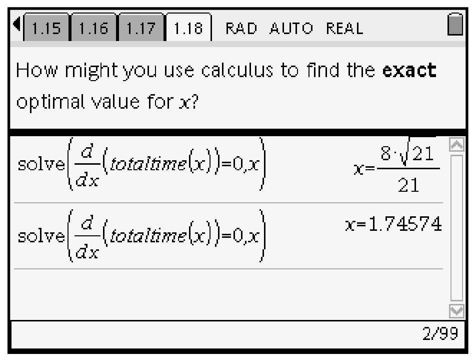
Step 4: Building and verifying an algebraic model

A need for greater precision should occur naturally from class discussion and give rise to the need for an algebraic model that can support predictions. Data is collected and displayed in a scatter plot, and an algebraic model may be found. The minimum value is then found using calculus.

Overlaying the graph of this function on top of a scatter plot of the data, the algebraic model is a powerful tool for verification of the geometric model.







Student .tns File Solutions

The start of the beach race is a buoy, located 4 kilometers offshore from one end of a 6-kilometer beach. The finish line lies at the opposite end of the beach.

"The Beach Race" [Answers appear in brackets].

1. What is the total length of the race if you swim for the shortest possible distance?
   [4 km (from buoy to beach) + 6 km (length of beach) = 10 km]
2. What is the total length of the race if you swim the *entire* distance?
   [$\sqrt{4^2+6^2}=\sqrt{52}\approx 7.2$ km]
3. Which of these strategies would take longer, assuming you swim at a rate of 4 km·h^{-1} and run at rate 10 km·h^{-1}? By how much longer?
   [Strategy 1 would take about 1.6 hours, and strategy 2 would take about 1.8 hours, so strategy 2 would take about 20 minutes longer.]
4. What landing point would produce the longest time for the race?
   [swimming directly to the finish line: 1.8 hours]

5. Use the scatter plot to approximate the value of x that leads to the minimum time. Then use calculus to find the exact value.
   [$x \approx 1.75$; $x = \dfrac{8\sqrt{21}}{21} \approx 1.74574$]
6. What are the main limitations of this model? What assumptions have you made in creating this model? [Some of the many limitations include the accuracy of the construction, as well as neglect of "reality" factors—maintaining the same speed under all conditions, swimming in perfectly straight lines, and perfect flatness of the beach are not realistic assumptions.]

Students should be expected to document their solutions carefully in their own words, detailing the meaning of each result and the reasons for their choices of approach. In particular, students should be able to clearly describe both the strengths and the weaknesses of their models.

Teachers should ensure that they recognize and reward elements of persistence, care in construction and measurement, and ability to justify reasoning in responses submitted for this task.

Extensions

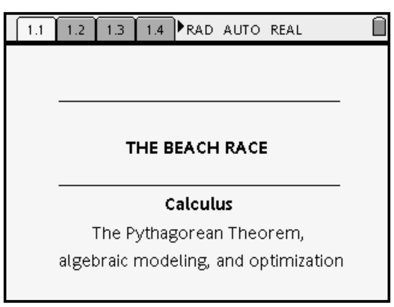
Students should consider other approaches that may be possible for this problem—in particular, what might be reasonable values for the transition between surf and sand? For what length of time would racers be somewhere between their swim speed and their run speed as they reach shore? How might this be built into a model?

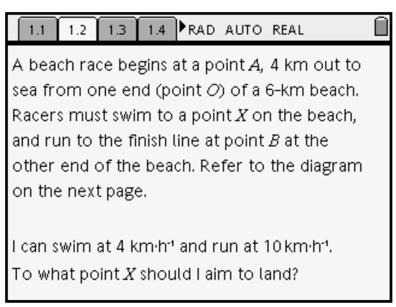
Is it possible to generalize the solution to this problem? Is there an optimal relationship between swim speed and run speed that might influence the result?

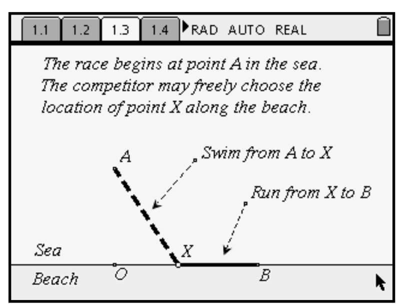
Other types of races could be examined in this way—for example, triathlons are popular events involving swimming, running, and cycling. A suitable extension assignment could involve students in modeling a triathlon event, using their own measurements and other sources (such as the Internet) to derive reasonable estimates of speeds for each leg.

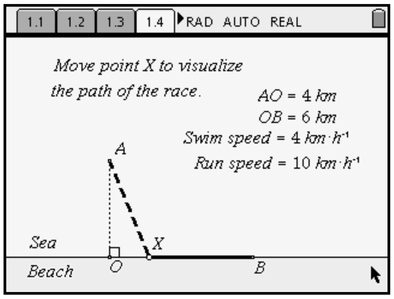
ACTIVITIES EXCHANGE

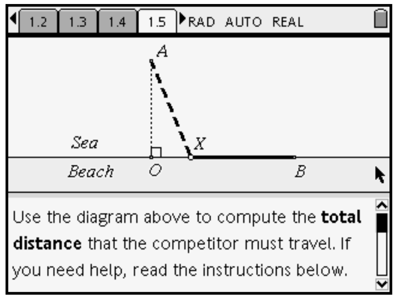
Visit education.ti.com/exchange to download activity files, including the student .tns file CalcAct13_BeachRace_EN.tns. Enter "8320" in the quick search box.

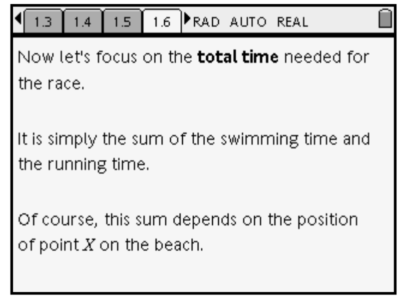


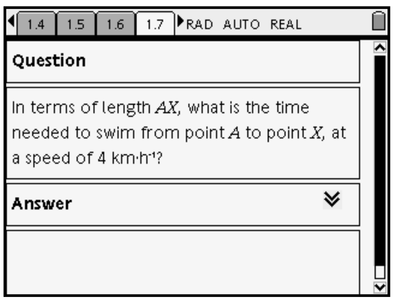


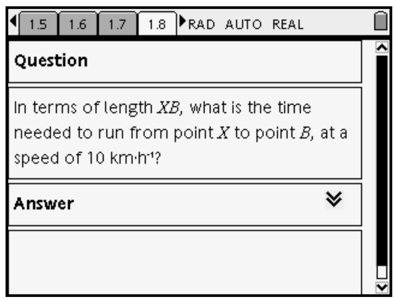


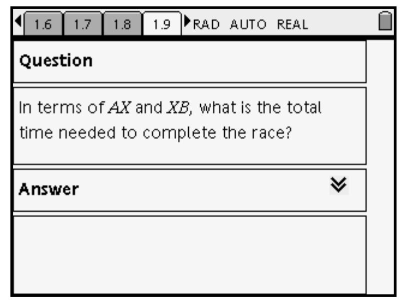

1.1 1.2 1.3 1.4 RAD AUTO REAL
Move point X to visualize the path of the race.
AO = 4 km
OB = 6 km
Swim speed = 4 km·h⁻¹
Run speed = 10 km·h⁻¹
A
Sea
X
Beach
O
B

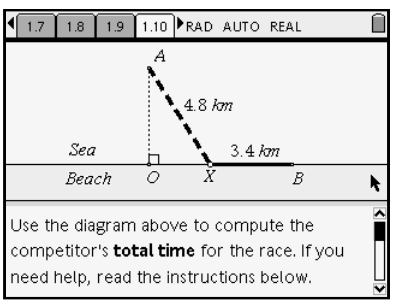

1.2 1.3 1.4 1.5 RAD AUTO REAL
A
Sea
X
Beach
O
B
Use the diagram above to compute the **total distance** that the competitor must travel. If you need help, read the instructions below.

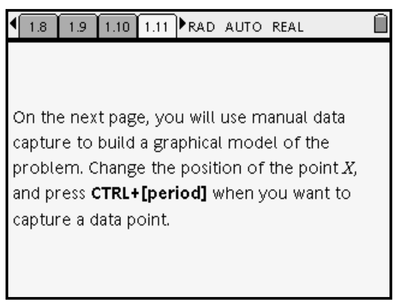

1.3 1.4 1.5 1.6 RAD AUTO REAL
Now let's focus on the **total time** needed for the race.
It is simply the sum of the swimming time and the running time.
Of course, this sum depends on the position of point X on the beach.

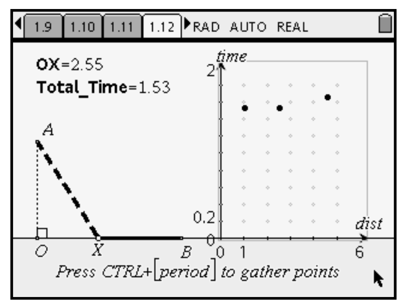

1.4 1.5 1.6 1.7 RAD AUTO REAL
Question
In terms of length AX, what is the time needed to swim from point A to point X, at a speed of 4 km·h⁻¹?
Answer

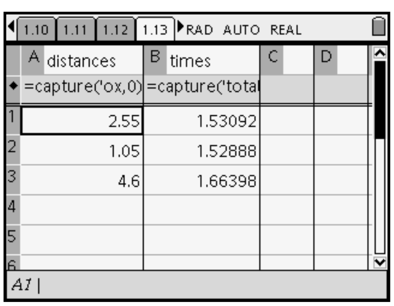

1.5 1.6 1.7 1.8 RAD AUTO REAL
Question
In terms of length XB, what is the time needed to run from point X to point B, at a speed of 10 km·h⁻¹?
Answer

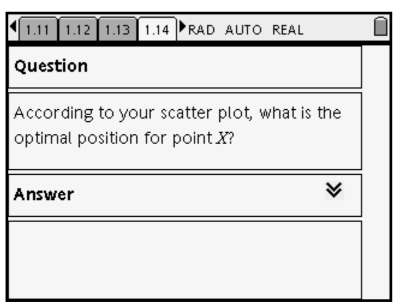

1.6 1.7 1.8 1.9 RAD AUTO REAL
Question
In terms of AX and XB, what is the total time needed to complete the race?
Answer

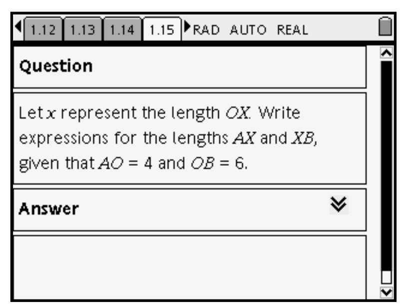

1.7 1.8 1.9 1.10 RAD AUTO REAL
A
4.8 km
Sea
3.4 km
Beach
O
X
B
Use the diagram above to compute the competitor's **total time** for the race. If you need help, read the instructions below.

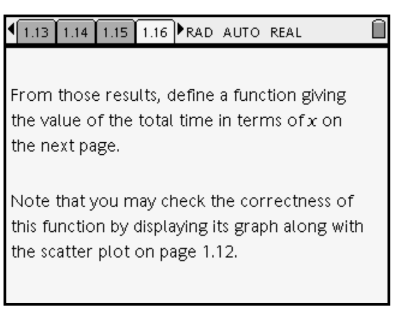

1.8 1.9 1.10 1.11 RAD AUTO REAL
On the next page, you will use manual data capture to build a graphical model of the problem. Change the position of the point X, and press **CTRL+[period]** when you want to capture a data point.

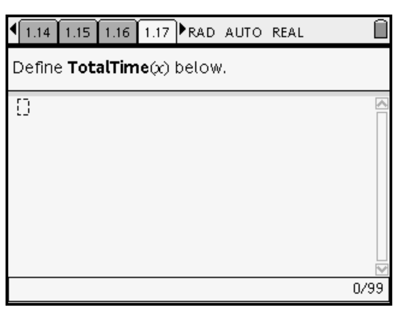

1.9 1.10 1.11 1.12 RAD AUTO REAL
OX=2.55
Total_Time=1.53
time
2
A
0.2
dist
O
X
B
0
0 1
6
Press CTRL+[period] to gather points

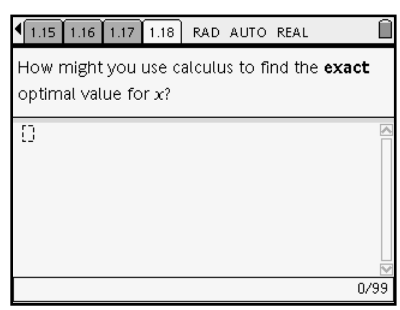

1.10 1.11 1.12 1.13 RAD AUTO REAL
A distances
B times
C
D
=capture('ox,0)
=capture('total
1 2.55 1.53092
2 1.05 1.52888
3 4.6 1.66398
4
5
6
A1 |


1.11 1.12 1.13 1.14 RAD AUTO REAL
Question
According to your scatter plot, what is the optimal position for point X?
Answer


1.12 1.13 1.14 1.15 RAD AUTO REAL
Question
Let x represent the length OX. Write expressions for the lengths AX and XB, given that AO = 4 and OB = 6.
Answer


1.13 1.14 1.15 1.16 RAD AUTO REAL
From those results, define a function giving the value of the total time in terms of x on the next page.
Note that you may check the correctness of this function by displaying its graph along with the scatter plot on page 1.12.


1.14 1.15 1.16 1.17 RAD AUTO REAL
Define **TotalTime**(x) below.
0/99


1.15 1.16 1.17 1.18 RAD AUTO REAL
How might you use calculus to find the **exact** optimal value for x?
0/99