AN INTRODUCTION TO SOLVING DIFFERENTIAL EQUATIONS

by C. C. Edwards
Coastal Carolina University, Conway, SC
Edwards@coastal.edu
(Please feel free to email me questions and /or comments.)

Key Topic: Differential Equations

Abstract:

In this activity we review the basic steps of solving differential equations, and illustrate these steps with an example. An example of an application of differential equations is also given. The students are then challenged to solve a few differential equations and applications of differential equations.

This activity is best used just after the student has been introduced to the concept of an integral.

Prerequisite Skills:

- Knowledge that integration is the inverse operation of differentiation
- Ability to integrate using the TI-89
- Ability to solve equations using a TI-89

Degree of Difficulty: Easy to moderate

Needed Materials: TI-89

NCTM Principles and Standards:

- Content Standards – Algebra
    - Represent and analyze mathematical situations and structures using algebraic symbols
    - Use mathematical models to represent and understand quantitative relationships
    - Draw a reasonable conclusion about situation being modeled

- Process Standards
    - Representation
    - Connections
    - Problem Solving

AN INTRODUCTION TO SOLVING DIFFERENTIAL EQUATIONS

A differential equation is an equation which involves derivatives. In its simplest form, these equations look like $\frac{dy}{dx} = f(x)$. To solve this equation for *y* as a function of *x*, you would do the inverse of differentiating, that is you would integrate. So the solution would be $y(x) = \int f(x)\,dx = F(x) + C$. But notice that because of the constant *C* of integration, this "solution" $y(x) = F(x) + C$ is really an infinite number of solutions. This family of curves $y(x) = F(x) + C$ is called the *general solution* to the differential equation $\frac{dy}{dx} = f(x)$. A *particular solution* to $\frac{dy}{dx} = f(x)$ is a single curve contained in the family of curves $y(x) = F(x) + C$ which constitute the general solution. Usually this particular curve is distinguished from the others in the family of solutions by identifying one point on the curve. When you substitute the coordinates of this point into the general solution $y(x) = F(x) + C$, you can then solve the resulting equation for *C*. Substituting that value of *C* back into the general solution $y(x) = F(x) + C$ will give you the particular solution.

These steps are summarized in the following simple example.

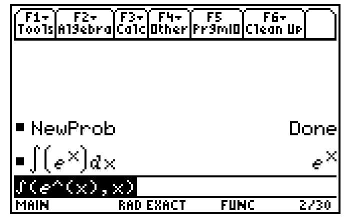
EXAMPLE 1. Find the solution to $\frac{dy}{dx} = e^x$ which contains the point $(0,\ 4)$.

Identify the differential equation	$\frac{dy}{dx} = e^x$
Find the general solution to this equation	$y(x) = \int e^x dx$ $y(x) = e^x + C$
Substitute the coordinates of the point into this solution and solve for *C*	$4 = e^0 + C$ $C = 3$
Substitute this solution for *C* into the general solution to find the particular solution.	$y(x) = e^x + 3$

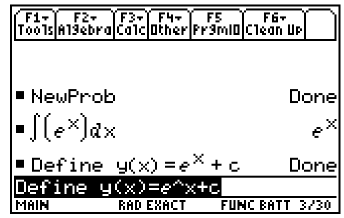
This problem was so simple that we didn't need any help from the TI-89 to solve it. But not all problems are this easy. So let's redo this problem using the TI-89 so that we can refresh your memory about the appropriate keystrokes to use to solve an integration problem and to solve an equation. Note that the TI-89 does give you the constant of integration. Since this is needed to find a particular solution, you must add it yourself.

***Whenever you start a new problem, clear memory by pressing* [2nd][F6]2.**

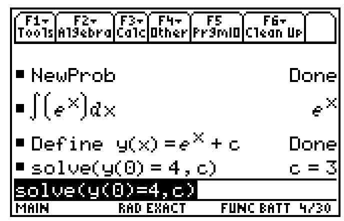
Enter and solve the integral by pressing [F3] [2], entering the function e^x, and then pressing [,] [X] [)] [ENTER].



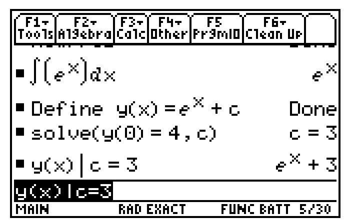
Define *y*(*x*) to equal this solution and add on the constant of integration. To do this press
[F4] [ENTER] [Y] [(] [X] [)] [=] [⏶] [ENTER] [+] [alpha] *c* [ENTER].



Solve this equation for *c* using the given point (0, 4).
To do this press
[F2] [ENTER] [Y] [(] [0] [)] [=] [4] [,] [alpha] *c* [)] [ENTER] .



To get the particular solution, evaluate the general solution *y*(*x*) with the new value for *c* by pressing
[Y] [(] [X] [)] [|] [⏶] [ENTER] [ENTER] .



***Whenever you start a new problem, clear memory by pressing* [2nd][F6]2.**

EXAMPLE 2. The rate of growth of a population of bacteria is proportional to the square root of time. The initial size of the population is 500. After 1 day, the population grows to 600. Approximately how large is the population after 7 days?

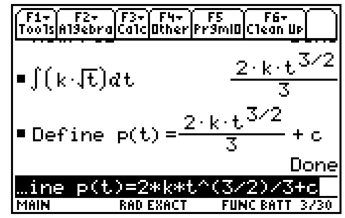
Variables: $t =$ time, measured in days
$p(t) =$ the size of the population after t days

Given: $p'(t) = k\sqrt{t}$ where k is the constant of proportionality.
$p(0) = 500$
$p(1) = 600$

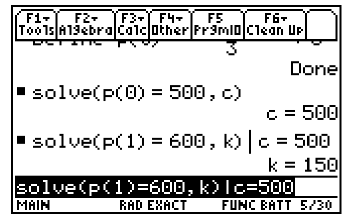
Find: Approximate $p(7)$

Solution:

Solve the differential equation and define *p*(*t*) to be this solution plus the constant of integration.

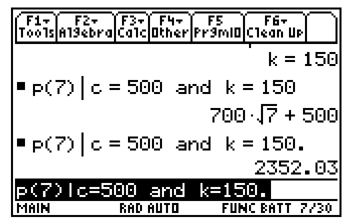


Use the fact that $p(0) = 500$ **to find *c*. Then use the fact that** $p(1) = 600$ **and the value you just found for *c* to solve for *k*.**



Evaluate *p*(7) using the values you found for *c* and *k*. The word "and" can be found by pressing [2nd] [MATH] [8] [8], or it can be typed in using the [alpha] key.

***p*(7) can be approximated by pressing [◆] [ENTER], or if you placed a decimal point after one of the numbers, the answer will automatically be approximated.**



The population is approximately 2352 bacteria after 7 days.

EXERCISES:

1. Find the general solution to the differential equation $y'(x) = e^x + \sin x$.

2. Find the solution to $y'(x) = e^x + \sin x$ which passes through the point $(0, 4)$.

3. Solve the second order differential equation $s''(t) = -32$ given that $s'(0) = 64$ and $s(0) = 80$.

   Hint: First solve $s''(t) = -32$ for the particular solution for $s'(t)$ which satisfies the given condition that $s'(0) = 64$.

4. An evergreen nursery usually sells a certain shrub after 6 years of growth and shaping. The growth rate during those 6 years is approximated by $\dfrac{dh}{dt} = 1.5t + 5$ where t is the time in years and h is the height in centimeters. If the seedlings are 12 centimeters tall when planted, how tall are the shrubs when they are sold?

ANSWERS:

1. $y(x) = e^x - \cos x + C$

2. $y(x) = e^x - \cos x + 4$

3. $s(t) = -16t^2 + 64t + 80$

4. 69 centimeters