Solving Systems of Linear Equations
With Row Reductions to Echelon Form
On Augmented Matrices

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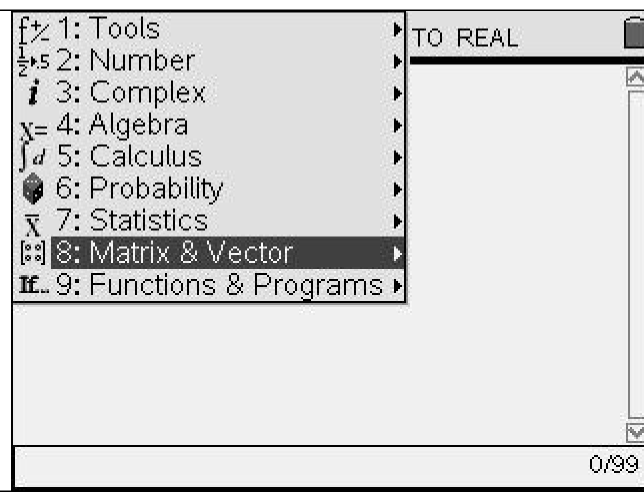
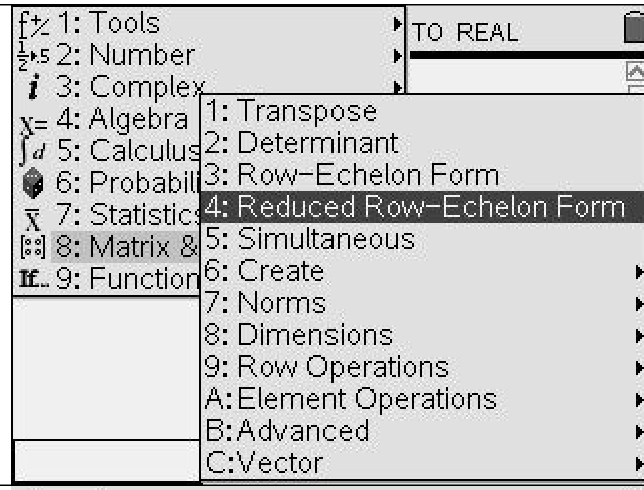
There is no more efficient way to solve a system of linear equations than with **R**ow **R**eduction to **E**chelon **F**orm (**rref**) on an augmented matrix. There are a variety of algebraic, geometric, and numerical methods available, but using the **rref** command applied to an input augmented matrix handles all possible cases of solving systems of linear equations. The parts of the following activity will show how to use the ***TI-Nspire*** to deal with a variety of systems, some of which have unique solutions, infinitely many solutions, or no solutions at all. (NOTE: The TI-83+/84 also have the RREF command at MATRIX > CALC > RREF to use on an input augmented matrix.)

To get started, look at the example of a system of 3 linear equations containing 3 variables/unknowns.

GIVEN the system of 3 linear equations in standard form: $\begin{cases} 5x+6y-4z=-4 \\ 4x-3y-5z=22 \\ 7x-y+6z=11 \end{cases}$

The coefficients and constants in the system are going to be entered as a 3 by 4 matrix, one row for each equation. The following screen shots and instructions were captured on a TI-Nspire CAS calculator, but the same keystrokes will work similarly on the regular TI-Nspire calculator as well.

On the ***TI-Nspire CAS***, open a document 1: Add Calculator page Choose menu 8: Matrix & Vector (NOTE: This is **option 7** on the ***Nspire***.)	

<table>
<tr>
<td>4: Reduced Row-Echelon Form to create an rref() command.</td>
<td>
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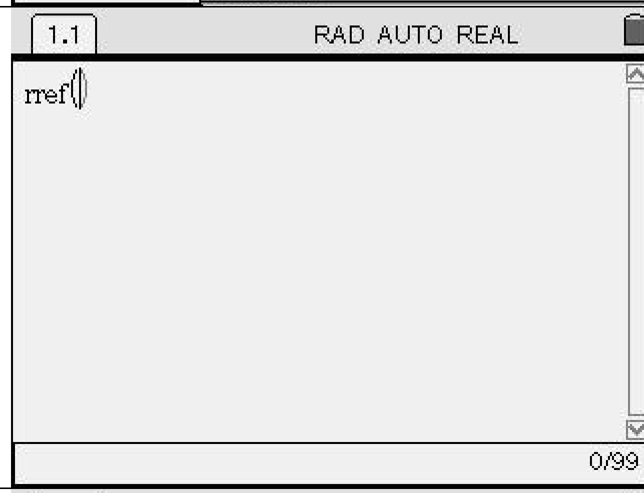
<td>With the cursor inside the parenthesis,.</td>
<td>
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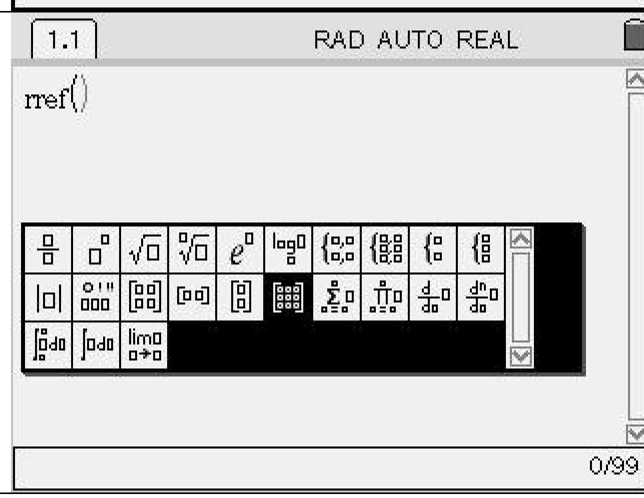
<td>Choose ctrl Matrix</td>
<td>
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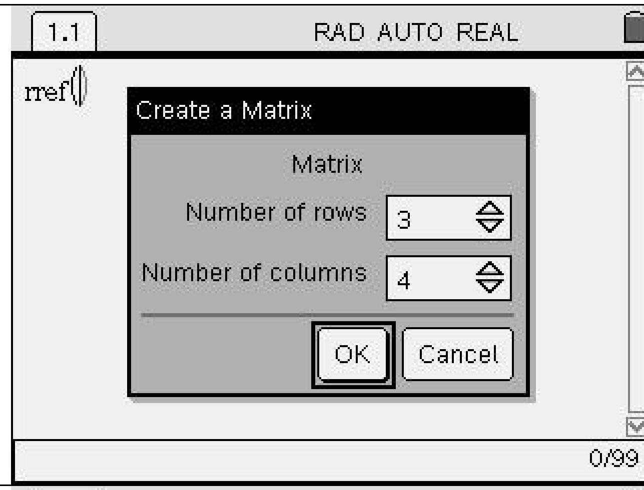
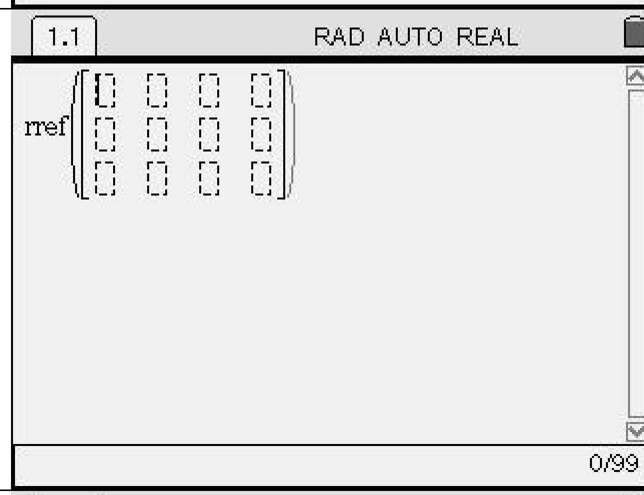
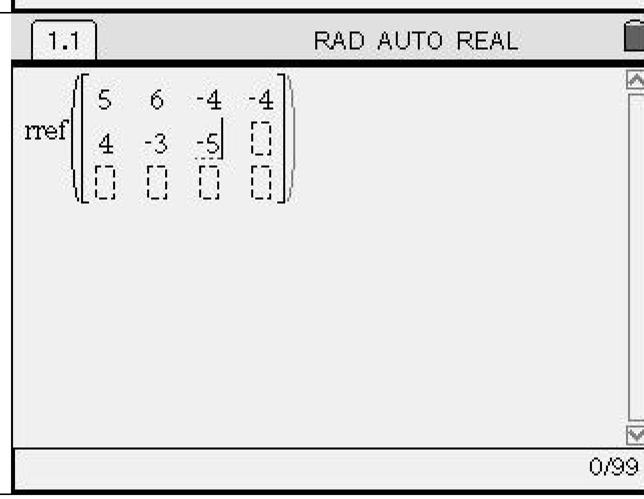
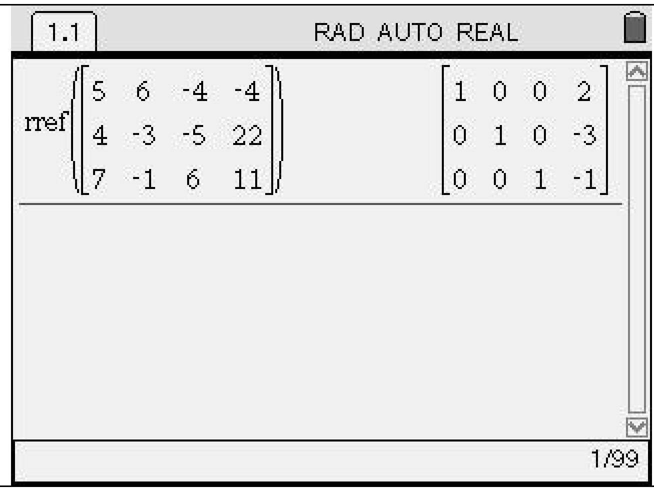
Select 3 rows and (tab) to enter 4 columns, then (tab) to **OK** to create the template to receive the values for this system	
The field of rows and columns are set to receive the coefficients and constants from the system in the example. NOTE: It is best to use the (tab) key to move from one entry field to the next.	
The (tab) key will move through the fields column by column and row by row. The arrow keys on the NAVPAD are not as automatic as the (tab) key for navigating the field.	

Use the enter is after value entry is complete. The matrix will automatically be row reduced to echelon form shown on the right as output.	

The output in the last screen shot translates into the following set of equations revealing the solutions to the system of linear equations for this example:

$$\begin{cases} 1x+0y+0z=2 \\ 0x+1y+0z=-3 \\ 0x+0y+1z=-1 \end{cases} \text{ or } \begin{cases} x=2 \\ y=-3 \\ z=-1 \end{cases}$$

It should be easy to see that the row reduced form gives the solution for the values of *x, y,* and *z* in the last column. Reduction to ***eschelon form*** indicates that the main diagonal is populated with 1's and elements off the diagonal are 0's when the system has a unique solution. However, two other possibilities can occur when RREF is applied to an input augmented matrix representing some system of equations.

The first alternative is a system that has no solutions. A reduced-echelon form that has a row of zeros in the coefficient section and a non-zero number in the augmentation column occurs with a system that has no solution. The following is an example of a system translated into an augmented matrix and the outcome that occurs when RREF is applied to the system. Follow the steps above for entering this example. Remember to use the tab key for moving through the entry fields of the matrix you created.

$$\begin{cases} 2x+y+2z=-12 \\ 4x+3y-3z=-2 \\ 6x+5y-8z=9 \end{cases} \text{ should be entered as } rref\left(\begin{bmatrix} 2 & 1 & 2 & -12 \\ 4 & 3 & -3 & -2 \\ 6 & 5 & -8 & 9 \end{bmatrix}\right) \text{, then press enter.}$$

The output should be $\begin{bmatrix} 1 & 0 & \frac{9}{2} & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The last row translates into the **false** statement

$0x+0y+0z=1$, indicating **no solution.**

<u>The second alternative is a system with infinitely many solutions</u>. The following is an example of a system translated into an augmented matrix and the outcome that occurs when RREF is applied to the system:

$\begin{cases} x+y-2z=-15 \\ 3x+y-z=-20 \\ 5x+y=-25 \end{cases}$ should be entered as $rref\left(\begin{bmatrix} 1 & 1 & -2 & -15 \\ 3 & 1 & -1 & -20 \\ 5 & 1 & 0 & -25 \end{bmatrix}\right)$, then press enter

The output should be $\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -\frac{5}{2} & -\frac{25}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The last row translates to the **true** statement $0x+0y+0z=0$ indicating infinitely many solutions.

In this second case, the variable *z* is considered a value open to selection or an *arbitrary variable*. However, the values of *x* and *y* vary along with the choice of value for variable *z*. Consider the analogy to the social process of one person becoming engaged to marry another person. With the choice of future spouse also comes the selection of future mother- and father in-law and a whole bunch of other in-laws, but the initial selection of a significant other is a more-or-less arbitrary choice. The same is true with choice of *z* and the subsequent determination of the values of *x* and *y*. To generate members of the solution set, consider the row-reduced form of the previous example:

$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -\frac{5}{2} & -\frac{25}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ translates back into the system $\begin{cases} x+\frac{1}{2}z=-\frac{5}{2} \\ y-\frac{5}{2}z=-\frac{25}{2} \\ z\, is\, arbitrary \end{cases}$ or $\begin{cases} x=-\frac{5}{2}-\frac{1}{2}z \\ y=-\frac{25}{2}+\frac{5}{2}z \\ z\, is\, arbitrary \end{cases}$

Here are some of the infinitely many solutions that exist for this system based on random choices of *z*. All of the solutions are generated by the equations for x and y:

z	***x***	***y***	**(*x*,*y*,*z*)**
0	$-\frac{5}{2}$	$-\frac{25}{2}$	$\left(-\frac{5}{2},-\frac{25}{2},0\right)$
1	3	-10	$(3,-10,1)$
9	2	10	$(2,10,9)$
-10	$\frac{5}{2}$	$\frac{25}{2}$	$\left(\frac{5}{2},\frac{25}{2},-10\right)$

EXAMPLES OF OTHER SYSTEMS OF EQUATIONS:
(NOTE: Refer back to the calculator entry instructions on the first page of this activity.)

Example #1: A system of two equations with two unknowns:

$$\begin{cases} 3x + y = -5 \\ 5x - 3y = -13 \end{cases}$$

On the TI-Nspire: $rref\left(\begin{bmatrix} 31 & -5 & 5 \\ -3 & -3 & -13 \end{bmatrix}\right)$ enter.

Output should be $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$, indicating the solutions are $x = -2$ and $y = 1$.

Example #2: Arrange the system of two equations with two unknowns into standard form (or solve the system by graphing the two lines to find their point of intersection!).

$$\begin{cases} y = \frac{5}{2}x - \frac{3}{2} \\ y = -\frac{3}{2}x + \frac{5}{2} \end{cases} \text{ restates as } \begin{cases} 5x - 2y = 3 \\ 3x + 2y = 5 \end{cases}.$$

On the TI-Nspire: $rref\left(\begin{bmatrix} 5 & -2 & 3 \\ 3 & 2 & 5 \end{bmatrix}\right)$ enter.

Output should be $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, so the solutions are $x = 1$ and $y = 1$.

Example #3: An extra equation is included in the system with two unknowns to be identified.

$$\begin{cases} 2x - y = -3 \\ x - 2y = -6 \\ 7x - 6y = -2 \end{cases}$$

On the TI-Nspire solve an augmented matrix choosing the first two of the equations given. This "overstocked" system has solution (4,5).

Remember that a solution to a system must make all equations in the system true, so check the solution from the two equations selected in the third equation. If the ordered pair fails in the 3rd equation, the system has no solution.

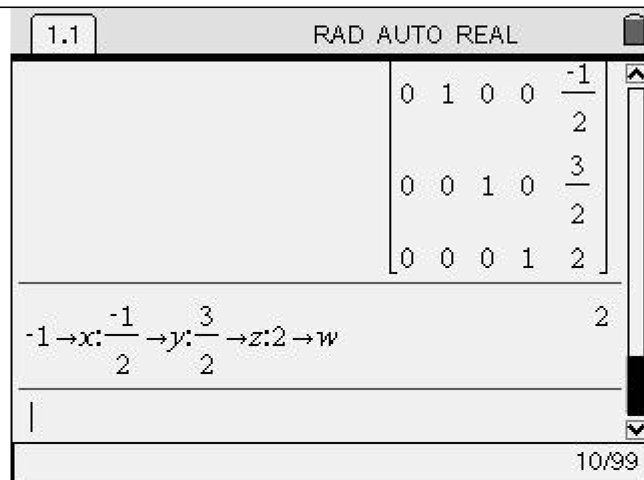
Example #4: A system of 4 equations with 4 unknowns, so your augmented matrix will have 4 rows and 5 columns.

$$\begin{cases} 3x-2y+4z+w=6 \\ x+4y-10z-w=-20 \\ -x-y+z+2w=7 \\ 6x+9y-z-3w=-18 \end{cases}$$ Solve with TI-Nspire: $rref\left(\begin{bmatrix} 3 & -2 & 4 & 1 & 6 \\ 1 & 4 & -10 & -1 & -20 \\ -1 & -1 & 1 & 2 & 7 \\ 6 & 9 & -1 & -3 & -18 \end{bmatrix}\right)$ enter.

Output should be $\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$, so the solutions are (x, y, z, w) = $\left(1, -\frac{1}{2}, \frac{3}{2}, 2\right)$.

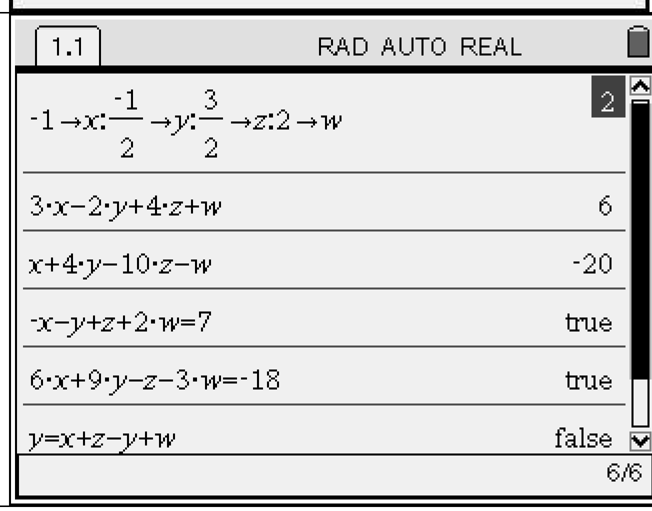
This is a good time to use the TI-Nspire to check the solution to the system numerically.

The ctrl sto→ var key is used to assign a numerical value to a variable from the system. The values can be stored one per entry line or altogether as shown with the STO commands separated by colons.



The left member of each equation is entered as it appears in the system followed by an enter key. The value of the equation should agree with the constant for that equation in the system.

You can also enter the entire equation and check for a ***true*** or ***false*** classification on the entry. This method would be particularly easier is cases where variables are present in both members of the equations.



Example #5: This system has only 3 equations, but 4 variables, so the system could have infinitely many solutions and an arbitrary variable or no solution at all.

$$\begin{cases} 2x+2y+z=11 \\ 2x+3z=5 \\ 3x-3y+2w=8 \end{cases}$$

On the TI-Nspire: $rref\left(\begin{bmatrix} 2 & 2 & 1 & 0 & 11 \\ 2 & 0 & 3 & 0 & 5 \\ 3 & -3 & 0 & 2 & 8 \end{bmatrix}\right)$ enter.

The output should be $\begin{bmatrix} 1 & 0 & 0 & \frac{2}{5} & \frac{22}{5} \\ 0 & 1 & 0 & -\frac{4}{15} & \frac{26}{15} \\ 0 & 0 & 1 & -\frac{4}{15} & -\frac{19}{15} \end{bmatrix}$

Notice that the output shows the matrix is reduced to echelon form through the first three columns. This is a pretty messy output in this example, but it contains the equations that generate all solutions to the system based on the arbitrary selection of a value for the variable w.

$\begin{cases} x+\frac{2w}{5}=\frac{22}{5} \\ y-\frac{4w}{15}=\frac{26}{15} \\ z-\frac{4w}{15}=-\frac{19}{15} \end{cases}$ restated to generate values of $x, y,$ and z using w's value: $\begin{cases} x=\frac{22}{5}-\frac{2w}{5} \\ y=\frac{26}{15}+\frac{4w}{15} \\ z=-\frac{19}{15}+\frac{4w}{15} \\ w\, is\, arbitrary \end{cases}$

SUMMARY: Row reduction operations on an augmented matrix are an effective method for solving an *n* X *m* system of linear equations using the *Nspire*'s **rref** command. The output is easily interpreted to give unique solutions, generate solutions based on an arbitrary variable, or indicate the system has no solution. When a correct solution is the goal rather than showing a particular method, row reducing the system to echelon form is the shortest path to that goal.