Approximating Gradient and Area

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Introduction

Differential and Integral Calculus in VCE Mathematical Methods U3&4 from 2023 onwards requires students to be familiar with techniques for approximating seemingly impossible calculations. These are the gradient of the tangent to a curve (What? Gradient requires two points, but a tangent line has only one point in common with the curve…!!) and area between a curve and the x-axis (Hold on!! How can you do that if the curve is a parabola, or a cubic? I don't know any area formulae for these shapes!!).

These queries are precisely why we need to have a way of getting *close* to the required calculation value. By learning how TI-Nspire CAS technology can assist us in this process, we should have a better appreciation for the power of Differential and Integral Calculus to find the *exact* gradient and area values.

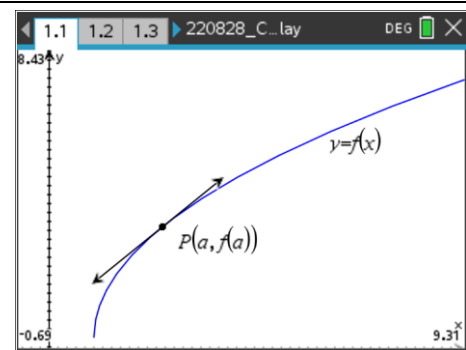## Central Difference Approximation – some theory

The problem:

Find the gradient of the tangent line to a graph of a function as shown opposite.

Note:

In assessment questions, you will work with a specific function (here, called $f(x)$) and a specific x-coordinate of P (here, called *a*). $P(a, f(a))$ is the point of tangency.


The task:

As gradient of a line requires two specific points, the Central Difference technique asks you to choose two points L (for **left** of P) and R (for **right** of P), nearby P on the graph of $f(x)$ which are **equidistant** (horizontally) from P. Thus you need to find (or, you'll be told to use) a suitable value for *b* which will be used to find the x-coordinates of $L(a-b, f(a-b))$ and $R(a+b, f(a+b))$.

The blue line, $\overrightarrow{LR}$ is a secant line as it intersects the graph of $y = f(x)$ twice. Here, note its gradient is close to that of the tangent line – they're not quite parallel.

Author: R. Wander

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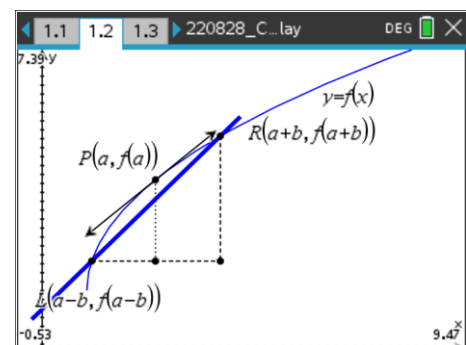
An important observation:

This diagram shows the same graph of $y = f(x)$, but the value of b is now smaller than it was in the previous diagram; so points L and R are closer to P, and the secant line's gradient appears to be much closer to that of the tangent line.
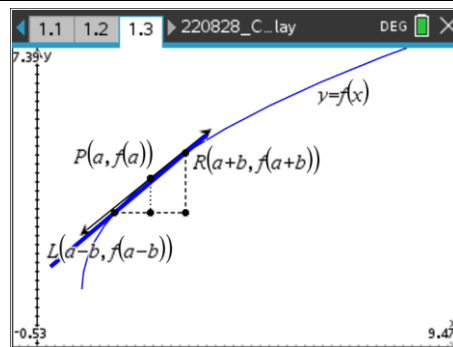
In this work, the gradient of the tangent line is written as $f'(x)$. We use the familiar gradient formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

with points L and R to write:

$$f'(x) \approx \frac{f(a+b) - f(a-b)}{(a+b) - (a-b)}$$

$$\approx \frac{f(a+b) - f(a-b)}{2b}$$

Question: 1.

As the value of *b* becomes smaller, the gradient of the secant becomes closer to the gradient of the tangent line, or curve, at *x=a*.

Question: 2.

Suppose you are asked to find the approximate value of the gradient to the curve: $f(x) = (x-2)\sqrt{x+1}$ at P where $x = 2$, using decreasing *b*-values 1 and 0.5. Complete the table below:

$f(x) = (x-2)\sqrt{x+1}$ and $a = 2$					
b	$a+b$	$f(a+b)$	$a-b$	$f(a-b)$	$f'(x) \approx \dfrac{f(a+b) - f(a-b)}{2b}$
1	3	2	1	$-\sqrt{2}$	$\dfrac{2+\sqrt{2}}{2}$
0.5	2.5	$0.5\sqrt{3.5}$	1.5	$-0.5\sqrt{2.5}$	$0.5(\sqrt{3.5} + \sqrt{2.5})$

Question: 3.

The graph of the function $f(x) = (x-2)\sqrt{x+1}$ is shown opposite. Locate point P where $a = 2$ and also the two points L and R for $b = 1$. Use a ruler to sketch the tangent at P in black and the secant $\overrightarrow{LR}$ in red.

It is known that the gradient of the tangent at P is exactly $\sqrt{3}$. Comment on the link between your sketch of the secant and the value you found using the Central Difference approximation technique.
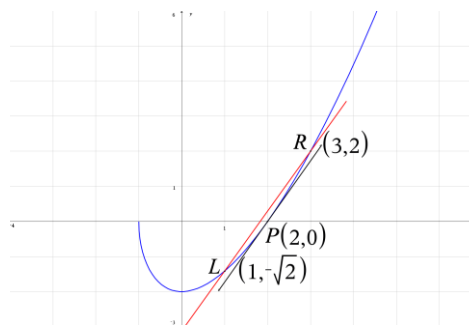
Answer: The secant looks parallel to the tangent, which is not surprising as the gradient of the secant $\overrightarrow{LR}$ is exactly $\dfrac{2+\sqrt{2}}{2}$ which correct to 3 d.p. is 1.707, which is close to the 3 d.p. approximate value of $\sqrt{3}$, which is 1.732

Author: R. Wander

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Central Difference Approximation – Using Nspire

We're now going to see how the calculations for secant gradients can be easily done on Nspire. We'll use the same function $f(x) = (x-2)\sqrt{x+1}$ with Point P located on the graph at $x = 2$. To do this we can use either a Notes page or a Lists & Spreadsheet page; shown below is the Notes page approach.

Open the file *cda1.tns* whose p1.1 is a Notes page.

Using **ctrl M** to open a Math Box, key in the function rule:
$f(x) = (x-2)\sqrt{x+1}$ then press **ctrl** and **var** then $f(x)$ to store this function; press **enter** and a second maths box will appear.
If the second maths box doesn't appear, press **ctrl M**.
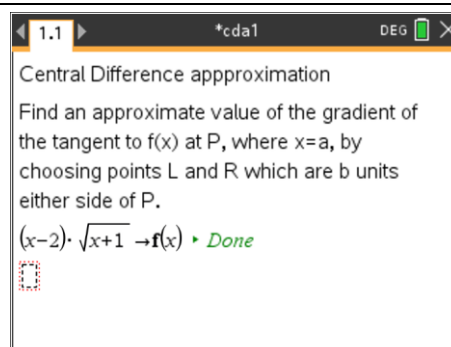
In the same way, store 2 →a and for now, store 1 →b. Press **enter** after each. Now key in the formula for the secant gradient
$$\frac{f(a+b) - f(a-b)}{2b}$$ and store it as the variable *sec_grad.*
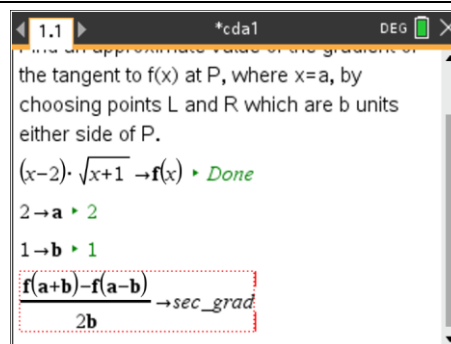
Question: 4.

When you press **enter** now, what do you notice?

Answer: You get the value we found in the first row of the table in Q2

Question: 5.

When you go into the Math Box for the variable *b* and change it to 0.5, what do you notice?

Answer: You get the value we found in the second row of the table in Q2

Question: 6.

Choose two more successively smaller values of *b* for your Notes page and comment.

Answer: Depends on values chosen, but if done correctly should yield values approaching $\sqrt{3}$

Question: 7.

(Extending) Now vary stored values of *a* or *b*, and/or for the function $f(x)$

Answer: Depends on values chosen. For functions with restricted domains, care will need to be taken near endpoint/s of that domain. Symmetric functions (say, quadratics when the b-value is the location of the turning point) will show all approximations will = 0 and graphical explanations are encouraged.

Teacher Notes:

The theory shown at the start of this activity is intended to complement that found in students' text or other teacher-provided resources. The function used in this general example is $f(x) = 2\sqrt{x-1} + 2$. The diagram at the bottom of p1 is easily constructed on a Graphs application using **menu Geometry** then **Points & Lines** , **Point On** etc; points on the graph can then be manipulated manually by the teacher in a classroom demo to illustrate the limiting ideas as the value of b decreases.

Choosing good examples of functions useful in illustrating the Central Difference approximation in the classroom is essential. If seeking to explore the gradient of a function in the neighbourhood of a turning point, where the "turning" is most obvious, the window settings should be adjusted so visually there's a clear gap between the tangent and the secant lines, and hopefully they will appear to be almost but not quite parallel.

The Notes page application is very useful for its instant updating based on new data (such as the function rule and the values of a and b) but no record of past attempts is "kept" by the software. Formulae on a Lists & Spreadsheet application, where successively smaller values of b can be entered row by row, provide a numerical path towards the limiting value of the gradient at the point of tangency. It was decided for the purpose of brevity not to show this approach in the associated webinar.

 Anyone seeking assistance in producing these Geometry-based diagrams within Graphs, or the Lists & Spreadsheet approach to the same examples seen here, is welcome to contact the author at rogerdwander@gmail.com or search the many webinars and tutorial videos provided on the website https://education.ti.com/en-au/professional-development/webinars-and-tutorials/live-webinars .

Author: R. Wander

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Trapezium Rule Approximation for Area – some theory

The problem:

Find the area between a curve defined as $y = f(x)$ and the *x*-axis and two vertical boundaries $x = a$ (in blue) and $x = b$ (in red), but we have no area formula for this shape.
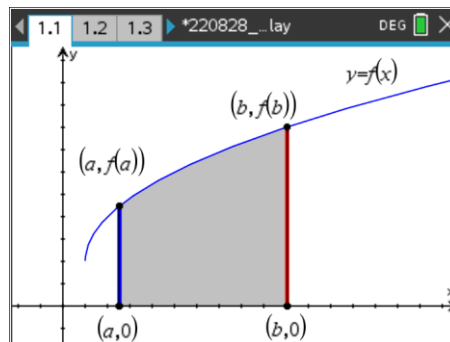
We can get close to this area by forming a trapezium.
In most textbooks, that formula appears as:

$$A = \frac{h(a+b)}{2}$$

It involves multiplying the *average* of the parallel side lengths (usually drawn horizontally) by the *height* (distance between those parallels. In our diagrams, the parallels are vertical; the blue with length $f(a)$ and the red with length $f(b)$, and we'll replace the previous "height" with the *width* which is $b - a$.

This trapezium area is approximately equal to the required area shaded in the previous graph.
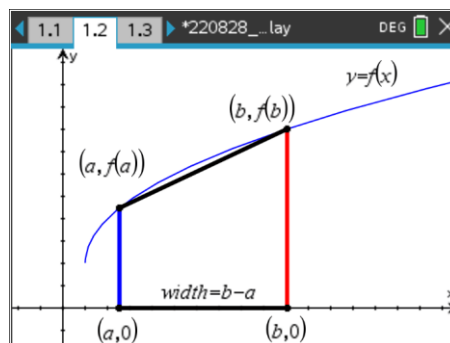
An even better approximation will be found if we form more than one trapezium between the boundaries. The width of each trapezium must be the same, and will be some fraction of the original $b - a$. You can see there's less "wastage" of area here, as the trapezia are much closer to the curve.
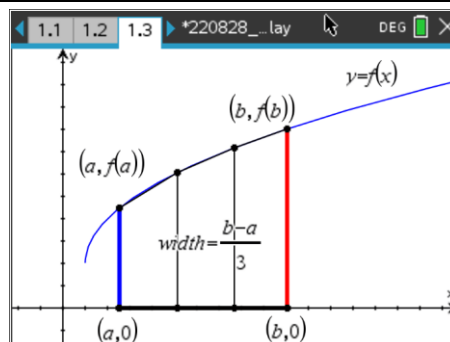
For Questions 8 and 9 below, you may wish to use your Nspire CAS to sketch the graph and perform calculations)

Question: 8.

Consider the curve given by $f(x) = (x-3)^2 + 2$. Suppose the approximate bounded area between the graph of $f(x)$, the *x*-axis and the lines $x = 2$ and $x = 5$ is required.

Find the width if 1 trapezium is used; 3 trapezia? 6 trapezia?

Area using 1 Trapezium:	3
Area using 3 Trapezia:	1
Area using 6 Trapezia:	0.5

Author: R. Wander

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Question: 9.

For the approximate area described in Question 8 above, the lengths of the left and right boundaries of the various trapezia need to be calculated. Suppose we're looking at using 3 trapezia between the lines $x = 2$ and $x = 5$. The first trapezium is between $x = 2$ and $x = 3$, so its left boundary has length $f(2) = 3$, and its right boundary is $f(3) = 2$. That trapezium has an area of $\dfrac{1 \times (3+2)}{2}$, or 2.5 sq units. Find the areas of the other two trapezia, and therefore the total approximate area.

Answer: Trap2 area is $\dfrac{1 \times (2+3)}{2}$ = 2.5, and Trap3 area is $\dfrac{1 \times (3+6)}{2}$ = 4.5; total is approx. 9.5 sq units; note integral calculus says area = 9 sq units exactly, thus an over-estimation (verify with graph).

Trapezium Rule Approximation for Area – Using Nspire

We're now going to see how the calculations for trapezium areas can be easily done on Nspire. We'll use the same function $f(x) = (x-3)^2 + 2$ with vertical boundaries located at $x = 2$ and $x = 5$. To do this we can use either a Notes application or a Lists & Spreadsheet application; shown below is the Notes application approach.

Open the file *areatrap1.tns* whose p1.1 is a Notes application.

As seen previously, press **ctrl M** to activate the Maths Box; into this key in the function rule $(x-3)^2 + 2$, store it using **ctrl var** as $f(x)$, and press **enter**.
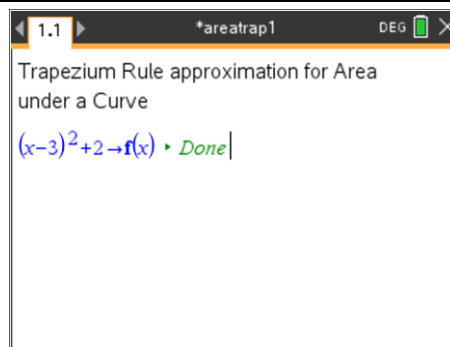
Similarly, store the values of $a = 2$ and $b = 5$ as shown.

We use *n* as the number of trapezia we wish to use; here, let's start with $n = 3$. Store that as well.

Since width (the fraction of the original $b - a$) is a crucial value for the area calculation, and referring to Q8, store $\dfrac{b-a}{n}$ as the variable *w*; press **enter,** this will allow the width to change as you change the values of *a*, *b* and *n*.
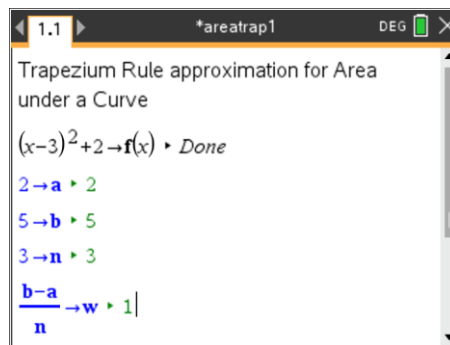
Before we get to the major formula, it's worth noting the diagram on the right for another quadratic function whose required area has been divided into 6 equally-spaced trapezia.

Notice here that $a = 1$, $b = 4$ and $n = 6$, so $w = \dfrac{4-1}{6} = \dfrac{1}{2}$.

Author: R. Wander

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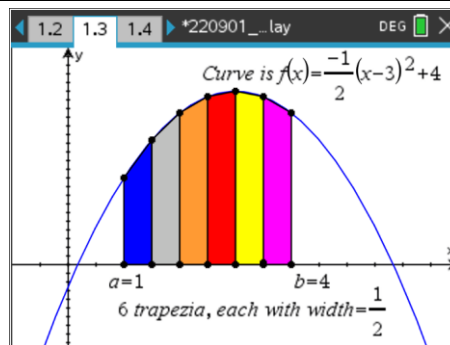
Let's look at the 3rd trapezium, called T(3). The *x*-coordinates of its vertical boundaries are $a+2w$ and $a+3w$, so the lengths of these boundaries are $f(a+2w)$ and $f(a+3w)$, respectively.

The area of T(3) is therefore $\dfrac{1}{2} \times \dfrac{f(a+2w)+f(a+3w)}{2}$;

the others can be expressed in a similar format.
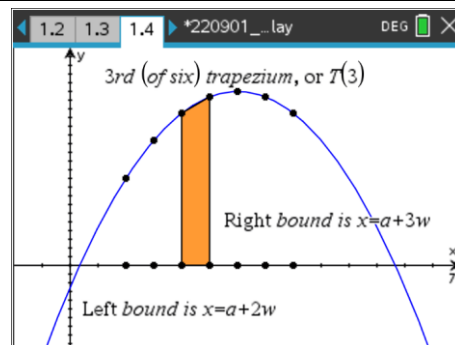
Now we're going to use Notes to store the formula for using these values to sum (or, add up) all the trapezia areas at once. The formula is:

$$\sum_{i=1}^{n} w \times \frac{f(a+(i-1)\times w)+f(a+i\times w)}{2}$$

Looks messy, but the summation sign indicates the values of i (1,2,3,4,5,6) are substituted into the formula, one by one, and the calculations are then added to produce the overall approximate area.

We go back to the Notes p1.1; with the cursor in a Math Box we press the template key: ⌨ (found to the right of the 9 key) and find the summation template. Carefully fill in details as shown – use multiplication signs where required.

When you press enter, the required area sum will be displayed.
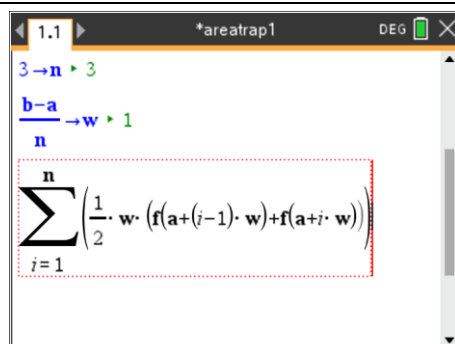
Question: 10.

How does your answer compare to that found in Q9 earlier?

Answer: It's the same because $\dfrac{19}{2} = 9.5$

Question: 11.

Now suppose we want to look at the approximate area with 6 trapezia, not 3. Alter the Math Box for the variable *n*, and discuss the meaning of your result compared with that found in Question 10. Use the fact that the actual exact area, in this case, is a whole number.

Answer: The new value is now $\dfrac{73}{8} = 9.125$, which is less than the previous answer. As the following graph shows, any approximation of this area using trapezia will be an over-estimation of the exact area.

Author: R. Wander

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Teacher Notes:

The theory shown at the start of this activity is intended to complement that found in students' text or other teacher-provided resources. The function used in this general example is $f(x) = 2\sqrt{x-1} + 2$. Knowing that some students may not easily adjust to the parallel sides of a trapezium being vertical, the author included a detailed explanation (mainly so students looking for "height" in the formula would be now comfortable with "width").

Choosing good examples of functions useful in illustrating the Trapezium approximation for area under a curve in the classroom is essential. It is suggested that the graph shows a clear gap between it and the *x*-axis, and that there's enough curvature to make the trapezia clearly over- or under-estimating the actual exact region as described.

The Notes page application is very useful for its instant updating based on new data (such as the function rule and the values of *a*, *b* and *n*) but no record of past attempts is "kept" by the software. It was decided to assume few students were familiar with the use of the summation notation template, hence the detailed explanation before using it in an example. Formulae on a Lists & Spreadsheet application, where each individual trapezium area is calculated (with the resulting sum also shown) , provide a numerical path towards the limiting value of the area sum as the number of trapezia increases. It was decided for the purpose of brevity not to show this approach in the associated webinar.

Anyone seeking assistance in producing these Geometry-based diagrams within Graphs, or the Lists & Spreadsheet approach to the same examples seen here, is welcome to contact the author at rogerdwander@gmail.com or search the many webinars and tutorial videos provided on the website https://education.ti.com/en-au/professional-development/webinars-and-tutorials/live-webinars .

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