Metallic Numbers

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Teacher Notes & Answers

7 8 **9** **10** **11** 12

TI-84PlusCE™ Investigation Student 30 min

Introduction

The famous Fibonacci sequence 1, 1, 2, 3, 5, 8 ... involves the recursive sequence definition: $t_{n+2} = t_n + t_{n+1}$.

The ratio between consecutive terms for the Fibonacci sequence as $n \to \infty$ is known as the Golden Ratio:

Golden Ratio: $\lim_{n\to\infty} \dfrac{t_{n+1}}{t_n} = \phi$

In this investigation you will explore a small variation on the Fibonacci sequence: $t_{n+2} = t_n + a t_{n+1}$ where a is a natural number. In this investigation these variants on the Fibonacci sequence will be referred to as "Levels", for example Fibonacci Level 2 means that $a = 2$ in the recursive definition above. The original Fibonacci sequence is therefore Fibonacci Level 1 with $a = 1$.

Teacher Notes:

Check out the "Curriculum Inspiration" on this activity via the Texas Instruments Australia website or YouTube. There are two versions, one for TI-Nspire and another for TI-84PlusCE.

https://education.ti.com/en-au/teachers

https://youtube.com/user/TIAustralia

Fibonacci Level 2: In search of the Silver Ratio

This sequence starts as: 1, 1, 3, 7, 17, 41, 99 …

Each successive term is equal to "the previous two terms plus another helping of the previous term." This can be expressed more succinctly using mathematical notation as:

$$t_{n+2} = t_n + 2t_{n+1}$$

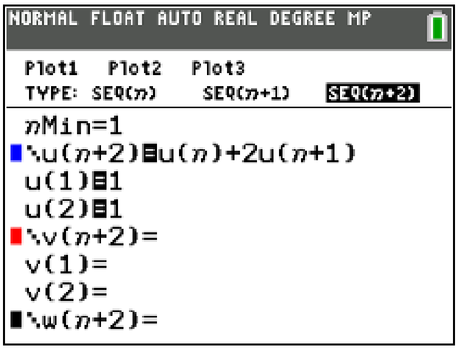
The first two numbers can still be set as 1 and 1.

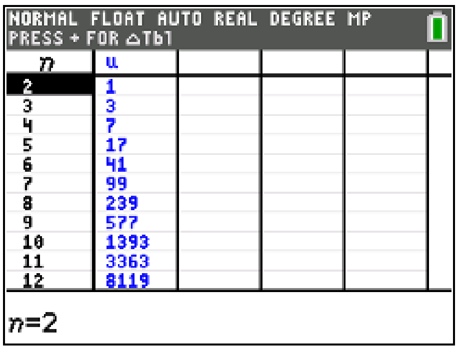
Make sure your calculator is set in 'sequence' mode: [MODE] > Seq

Use the y = editor to select the sequence type: SEQ(*n*+2)) shown.

Use [ALPHA] and 'u' (located above the 7 key) to reference terms.

To check the sequence, generate a table of values: [2nd] [GRAPH]





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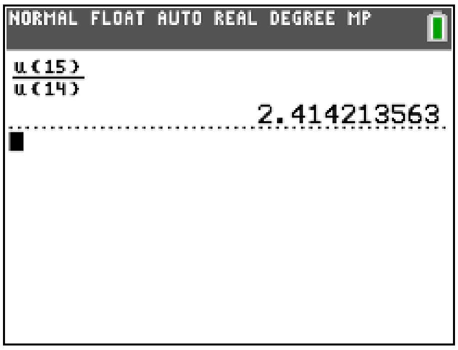
Calculate the ratio between consecutive terms.

Use the ‘u’ (located above the 7 key) to reference each term.



Question: 1.

Explore the ratio between consecutive Fibonacci Level 2 terms, this is called the ‘Silver’ ratio.

Answers will vary with regards to exploration. Example $L_1(11) / L_1(10) \approx 2.4142139267767$. Students may already start to recognise this value as $\approx 1+\sqrt{2}$. $L_1(21) / L_1(20) \approx$ 2.4142135623731

Calculator Tip!

It is possible to generate an entire list of ratios directly on the calculator home screen. Start by storing the numbers from 1 to 15 in L_1. The Sequence command located in the List > Op menu will help.

seq(x,x,1,15) $\rightarrow L_1$

With the Fibonacci Level 2 sequence defined in *u*, generate an entire list of ratios:

$$\frac{u(L_1+1)}{u(L_1)}$$

Question: 2.

Change the first two terms in the Fibonacci Level 2 sequence and check to see if this changes the long term value of the ratio between consecutive terms.

The common ratio still holds, (assuming both initial terms are non-zero) however the number of terms required to find the common ratio may change.

Question: 3.

Let x represent any term in the sequence and y the next term.

a) Explain the two formulas below:

$$r_n = \frac{y}{x} \quad \text{and} \quad r_{n+1} = \frac{2y+x}{y}$$

The first equation represents the ratio between consecutive terms according to how x and y have been defined above. The second equation simply follows the recursive definition for the level 2 Fibonacci sequence.

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b) Assuming the ratio between consecutive terms is approximately equal as $n \to \infty$ determine the value of the ratio.

There are several ways this problem may be tackled. Initially students may assume that with two equations and three unknowns, it is not possible to determine a specific value for the ratio, in this case however it is possible.

$xr_n = y$ (Eqn 1)

$r_{n+1} = \dfrac{2xr_n + x}{xr_n}$ (Sub. Eqn 1 into Eqn 2)

$r_{n+1} = \dfrac{2r_n + 1}{r_n}$ $(x \neq 0)$

$r^2 - 2r - 1 = 0$ $(r_{n+1} \approx r_n)$

$\left(r - 1 - \sqrt{2}\right)\left(r - 1 + \sqrt{2}\right) = 0$

$r = 1 + \sqrt{2}$ $r > 0$

$\dfrac{y}{x} = \dfrac{2y + x}{y}$ $(r_{n+1} \approx r_n)$

$y^2 = 2xy + x^2$

$y^2 - 2xy - x^2 = 0$

$(y - x)^2 - 2x^2 = 0$

$\left(y - x - \sqrt{2}x\right)\left(y - x + \sqrt{2}x\right) = 0$

$y = \left(1 \mp \sqrt{2}\right)x$ (Sub into Eqn 1)

$r = 1 + \sqrt{2}$ $(r > 0)$

Fibonacci Level 3: In search of the Bronze Ratio

The bronze ratio refers to the ratio between consecutive terms of the level 3 Fibonacci sequence. The general formula for the sequence $t_{n+2} = t_n + at_{n+1}$ therefore becomes: $t_{n+2} = t_n + 3t_{n+1}$

Question: 4.

Generate the first 50 terms of the level 3 sequence, store them in L_3 and explore the ratio between consecutive terms as n increases.

Content generated on the calculator. Students should explore the ratio using a similar technique to that used in Questions 1 and 2.

With $t_n = 1$ & $t_{n+1} = 1$

$L_3(11) / L_3(10) \approx 3.3027756406463$

$L_3(21) / L_3(20) \approx 3.302775637732$

These two examples show that the ratio appears to once again be approaching a limit. Changing the initial terms doesn't make any difference to the eventual limit.

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Question: 5.

Set up two formulas similar to those from Question 3 and hence determine the exact value for the bronze ratio.

$r_n = \frac{y}{x}$ (Eqn 1) $\qquad r_{n+1} = \frac{3y+x}{y}$ (Eqn 2)

$xr_n = y$ (Eqn 1#)

$r_{n+1} = \frac{3xr_n + x}{xr_n}$ (Sub. Eqn 1# into Eqn 2)

$r_{n+1} = \frac{3r_n + 1}{r_n}$ $\quad (x \neq 0)$

$r^2 - 3r - 1 = 0$ $\quad (r_{n+1} \approx r_n)$

$$\left(r - \frac{3}{2} - \frac{\sqrt{13}}{2}\right)\left(r - \frac{3}{2} + \frac{\sqrt{13}}{2}\right) = 0$$

$r = \frac{3+\sqrt{13}}{2}$ $\quad r > 0$

Fibonacci Level *n*: The Metallic Ratios

The general term for the ratio between consecutive terms for $t_{n+2} = t_n + at_{n+1}$ is referred to as a Metallic ratio.

Question: 6.

Determine an expression for the general form of the Metallic ratios. Check your answer using $a = 1$, $a = 2$ and $a = 3$.

$r_n = \frac{y}{x}$ (Eqn 1) $\qquad r_{n+1} = \frac{ay+x}{y}$ (Eqn 2)

$xr_n = y$ (Eqn 1#)

$r_{n+1} = \frac{axr_n + x}{xr_n}$ (Sub. Eqn 1# into Eqn 2)

$r_{n+1} = \frac{ar_n + 1}{r_n}$ $\quad (x \neq 0)$

$r^2 - ar - 1 = 0$ $\quad (r_{n+1} \approx r_n)$

$$\left(r - \frac{a}{2}\right)^2 - \left(\frac{a^2+4}{4}\right) = 0$$

$$\left(r - \frac{a}{2} - \frac{\sqrt{a^2+4}}{2}\right)\left(r - \frac{a}{2} + \frac{\sqrt{a^2+4}}{2}\right) = 0$$

$r = \frac{a+\sqrt{a^2+4}}{2}$ $\quad r > 0$

By Substitution for a = 1 the golden ratio: $r = \frac{1+\sqrt{5}}{2}$ and the silver ratio a = 2, $r = \frac{2+\sqrt{8}}{2} = 1+\sqrt{2}$ ...

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Question: 7.

For the golden ratio (ϕ) the following relationships hold:

$\phi = \frac{1}{\phi} + 1$	$\phi^2 = \phi + 1$	$\phi^1 + \phi^2 = \phi^3$

Do any of the above relationships hold for the silver or bronze ratio?

Students may use the specific ratios or go straight to the general case. Students should also note that the all three of the above relationships are the same and therefore if the silver or bronze relationship hold for the first relationship it would therefore hold for the second and third.

$$r = \frac{a + \sqrt{a^2 + 4}}{2}$$

$$\phi^2 = \phi + 1$$

$$\left(\frac{a + \sqrt{a^2 + 4}}{2}\right)^2 = \frac{\left(a + \sqrt{a^2 + 4}\right)}{2} + 1$$

$$\frac{a\sqrt{a^2 + 4}}{2} + \frac{a^2}{2} + 1 = \frac{a + \sqrt{a^2 + 4}}{2} + 1$$

$$(a-1)\sqrt{a^2 + 4} + a^2 - a = 0$$

$$(a-1)\sqrt{a^2 + 4} + a(a-1) = 0$$

$$(a-1)\left(\sqrt{a^2 + 4} + a\right) = 0$$

$$a = 1 \qquad\qquad \sqrt{a^2 + 4} + a > 0$$

The only valid solution for the relationships occurs when a = 1 (The Golden Ratio)

Question: 8.

Calculate the approximate value for each of the following and comment on your finding as the quantity of 'embedded' fractions increases.

a) $1 + \frac{1}{1+1}$ Answer = 1.5

b) $1 + \cfrac{1}{1 + \cfrac{1}{1+1}}$ Answer = $1.\dot{6}$

c) $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+1}}}$ Answer = 1.6

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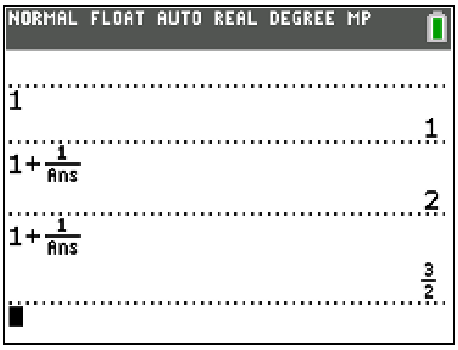
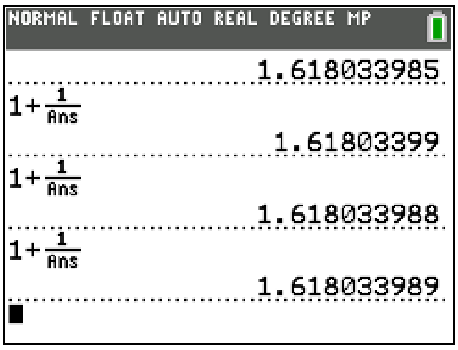
d) $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}}$ (In this case add as many fractions as possible)

Answers will vary depending on how many embedded fractions students used. Students can use the 'answer' (ans) functionality on the calculator to create this recursive computation. The screen (below left) shows that entering '1' as the first value automatically stores this in the most recent answer.

Expressing the recursive formula using 'ans' (2nd (-)) and repeatedly pressing enter makes this calculation very quick. The screen (below right) shows what happens after executing the command at least 20 times, students can see the convergence.





Students should recognise that this recursive fraction is approaching the golden ratio. This can be shown algebraically:

Let $x=1+\frac{1}{x}$

To show how this relates to the above, consider $x_n=1+\frac{1}{x}$ substituting x_n into x.

The above substitution results in: $x=1+\cfrac{1}{1+\cfrac{1}{x}}$ which can then be substituted again ... and again.

But $x=1+\frac{1}{x}$ can be expressed as $x^2-x-1=0$ the solution for which is Φ.

Question: 9.

Calculate the approximate value for the following 'embedded' fraction and comment on the result.

$$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}}}$$

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Students should recognise the result as the silver ratio and hypothesis the general case for the metallic ratios:

$$a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\ldots}}}}}$$

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