# Chapter 5

# Geometry and Measurement

## In this chapter

*This chapter summarizes NCTM's **Principles and Standards for School Mathematics** for geometry and measurement and applies those principles in some activity overviews. You will learn how to:*

- *Explore relationships between units of measure and objects to be measured.*
- *Develop your understanding of length, area, and volume.*
- *Help students develop spatial sense.*

## Overview of geometry and measurement

In the document *Principles and Standards for School Mathematics*, geometry and measurement are listed as separate content strands. Because measurement is the assignment of a numerical value to some attribute, measurement has connections to many other strands. Because many attributes that children measure have physical characteristics, geometry and measurement are frequently taught simultaneously.

### Goals for students

The NCTM measurement standards are stated as:

> Instructional programs from prekindergarten through grade 12 should enable all students to
>
> - understand measurable attributes objects and the units, systems, and processes of measurement;
> - apply appropriate techniques, tools, and formulas to determine measurements. (NCTM, 44)



Measurement lends itself to concrete objects and making comparisons between objects. However, children must expand their understanding and use of measurement to include relationships between the attributes being measured. In this manner there can be a logical connection to geometry from a standpoint of motion and change. One of the more simple relationships to demonstrate is that doubling the length of each side of a rectangle results in a rectangle four times as large in area. Children must be given the chance to explore such relationships. Teachers have a responsibility to assist students in making the connection between the concrete and the symbolic relationships explored as well as making connections to area and volume formulas.

Geometric ideas are often connected to other aspects of mathematics. For example, geometric pictures or models are frequently needed to solve a problem. Through the study of geometry, children should learn about motion and change as well as structure and shape. Spatial reasoning, especially in light of transformations of objects, is an important aspect of geometric learning. Geometry is also a natural area in which conjectures can be made, tested, and justified.

The study of geometry should build from informal to more formal thinking through the grades. You will see that technology can play an important part in the study of geometric relationships and principles. The NCTM geometry standards are stated as:

> Instructional programs from prekindergarten through grade 12 should enable all students to
>
> - analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;
> - specify locations and describe spatial relationships using coordinate geometry and other representational systems;
> - apply transformations and use symmetry to analyze mathematical situations;
> - use visualization, spatial reasoning, and geometric modeling to solve problems. (NCTM, 41)



## *Goals for teachers*

When discussing the components of a geometry curriculum for prospective elementary teachers, the CBMS points out the following key ideas. Even though prospective teachers have had a high school geometry course, much of the suggested content to be learned is new. With that in mind, the CBMS suggests teachers must develop competence in understanding length, area, and volume.

> A first goal in a geometry course for prospective teachers is the development of visualization skills – building and manipulating mental representations of two- and three-dimensional objects and perceiving objects from different perspectives. In exercises designed to cultivate these skills, teachers handle physical objects: build structures with cubes, create two-dimensional representations of three-dimensional objects, cut out and paste shapes. …With regard to length, area, and volume, teachers should know what is meant by one, two and three dimensions. …An understanding of the volume of a rectangular solid involves seeing the relationship between layers of three-dimensional units and the area of its base. Formulas for the area and volume of some other kinds of objects can build from an understanding of rectangles and rectangular solids. The study

> of rectangles and rectangular solids also leads to an understanding of how length, area, and volume scale under uniform dilation" (CBMS, 22).

Prospective teachers of middle school mathematics should build upon the ideas mentioned above so that they are able to focus student attention on properties of shapes as well as to provide opportunities for conjecturing and reasoning about geometry objects. The CBMS states:

> Prospective teachers need experiences that help them better understand the role of units of measurement, choose appropriate tools and methods for measuring, and recognize the complexity of relationships between different types of measure. Exploring area and perimeter by holding one measurement constant is such an experience. Other learning situations could involve scale changes in planes and in space, leading first to problems involving proportional reasoning and measurements of similar figures, and then to the meaning of congruence of both two- and three-dimensional space (CBMS, 34).

## *The use of calculators in teaching and learning geometry and measurement*

A calculator can be used in many ways when exploring problems with geometry and measurement. From computing to graphing, the calculator can provide students with a tool of value. Geometry and measurement problems often involve numbers and operations, patterns and functions, algebraic reasoning, or data collection, areas in which a calculator can be useful.

## *Sample Activities*

The activities below will help teachers learn mathematics, reflect on what they have learned, and consider how to teach these same ideas to elementary school students.

### *Activity 1: Popcorn Container*

Even everyday shapes like cylinders can generate interesting discussions.

Ming and Janie were looking at an 8.5-inch x 11-inch rectangular piece of paper and were considering how to use it to make a cylinder. Janie saw that she could roll the piece of paper and tape the opposite long edges together to make a cylinder without the bases. Ming saw the paper could also be used to form a different cylinder by rolling the paper and taping its opposite short sides. Janie and Ming then wondered whether these two cylinders would hold the same amount of unpopped popcorn.

Jamie and Ming recalled the formula for circumference of a circle, knowing the radius or diameter ($C = 2r$ and $C = \pi d$), as well as the formula for the volume of a cylinder ($V$ = area of base times height). Armed with this information and a calculator, they began the calculations necessary to answer the question of whether the two cylinders would hold the same amount of unpopped popcorn.

Answer questions 1 - 4 as a *learner* of mathematics.

**1.** Do you think the two cylinders would hold the same amount of unpopped popcorn?

**2.** Are you able to visualize rolling the paper in the ways mentioned to make the cylinders?

**3.** How do you think you can determine the solution to the problem?

**4.** Would the size of the piece of paper make a difference?

Answer questions 5 - 8 as a *teacher* of mathematics.

**5.** What use can be made of a calculator to solve this problem?

**6.** Why might children think that the cylinders would hold the same amount? Children need practice in examining situations in which areas change but perimeters stay the same.

**7.** What situations can you provide for children so they think about what stays the same and what varies?

**8.** What convincing argument can be made that would tell which, if either, cylinder would hold the most?

### *Answers and Comments*

Questions **1 - 4**

Although the solution often surprises people, the two cylinders do not have the same volume, even though they do have the same surface area. With only a square piece of paper, would the two cylinders have the same volume? The volume of Janie's cylinder is $(11 \div 2\pi)^2 \times \pi \times 8.5$, while Ming's is $(8.5 \div 2\pi)^2 \times \pi \times 11$.

Questions **5 – 8**

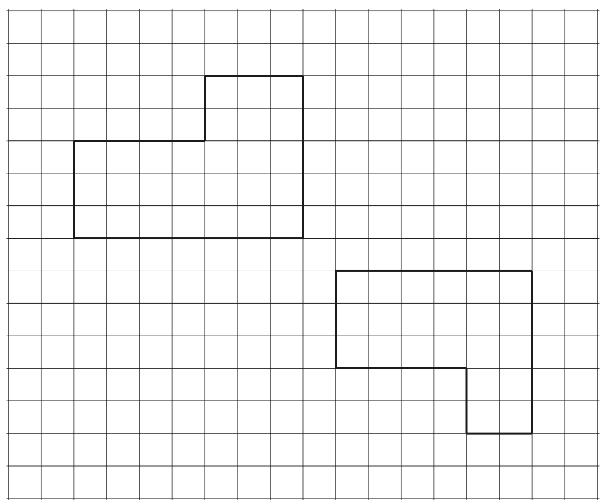
Although many people tend to think that the two cylinders would hold the same amount of unpopped popcorn because they are made from the same sheet of paper, this is not the case. A quick way to demonstrate this is to make each cylinder and place one inside of the other and fill the inner one. Carefully removing the inner cylinder and letting the popcorn begin to fill the outer cylinder usually brings some "ahs" from those watching who notice the outer cylinder is only about three-fourths filled. Use a calculator to perform the calculations necessary to find the volume of each container using the formula for the volume of a cylinder.

## Acitvity 2: Hexaright

(Adapted with permission from *Measuring Up - Prototypes for Mathematics Assessment*. Copyright 1993 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington, D.C.)

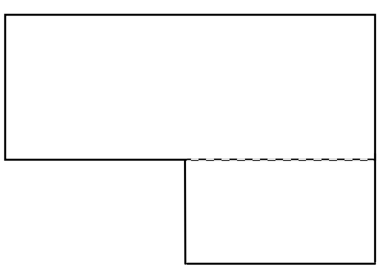
Children need practice with differentiating area and perimeter. They also need to understand that objects with the same perimeter may have different areas, and objects with the same area may have different perimeters. The following activity provides an opportunity for such an investigation.

Carol and Arsalan were drawing six-sided polygons (or hexagons) on square centimeter graph paper. (The graph paper below is not to scale.) The sides of each polygon were drawn along the lines on the graph paper. That is, adjacent sides were perpendicular.

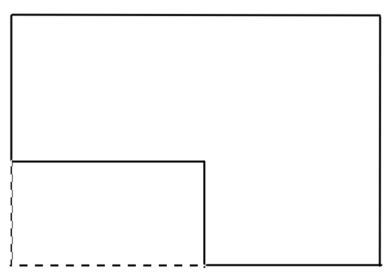


Each time they completed a figure, Carol and Arsalan found the perimeters as well as the areas. Sometimes one of them would have the largest area, but not the largest perimeter. This caused them to ponder. They decided to make as many different hexagons as they could with perimeters of 28 cm each.

To help with their calculations, Carol and Arsalan noticed that each time they drew a hexagon they could either partition it into two non-overlapping rectangles or view it as a rectangle with a rectangle removed from one corner. They used their calculators and the formula for the area of a rectangle to quickly determine the area of each hexagon they made. As they worked they began to wonder what would be the hexagon with the largest area they could find. As you answer the following questions, see if you can determine that hexagon.



Hexagon with two non-overlapping rectangles



Hexagon with a rectangle removed from one corner

Answer questions 1 - 6 as a *learner* of mathematics.

**1.** What difficulties might children have in recognizing these shapes as hexagons?

**2.** Using graph paper, make several hexagons with a perimeter of 28 cm. If your graph paper does not have square centimeters, use perimeters of 28 units and answer all questions with units in mind. How do the areas compare for these shapes? Are the areas different for each shape?

**3.** How could you find the perimeter of these hexagons without adding the lengths of all the sides? How is finding the perimeter of these hexagons like finding the perimeter of a rectangle?

**4.** What are some different methods for finding the area of each hexagon?

**5.** What was the area of the largest hexagon you could find with a perimeter of 28 cm? What do you think would be the area of the largest hexagon with a perimeter of 52 cm?

**6.** While following the lines of the graph paper, can you make a hexagon with a perimeter of 25?

Answer questions 7 - 12 as a *teacher* of mathematics.

**7.** How is a calculator useful when investigating this problem?

**8.** How would you help the children make a hexagon of perimeter 28 cm?

**9.** Does the use of graph paper provide guidance or would you think unlined paper could be used?

**10.** What other ways would you try to have children distinguish between area and perimeter?

**11.** What are the advantages and disadvantages of using a hexagon instead of a rectangle to investigate this question?

**12.** Is there anything special about using the numbers 28 cm and 52 cm to investigate this problem?

### *Answers and Comments*

Questions **1 - 4**

Often students are not used to seeing hexagons that are not convex. One way to view these hexagons is as a rectangle with a rectangular piece removed. Interestingly, the perimeter of the rectangle and its resulting hexagon will be the same; so one way to find the perimeter of these hexagons is to find the perimeter of the rectangle without the piece removed.

Question **5**

There are numerous hexagons with a perimeter of 28cm that have different areas. If the perimeter follows the lines of the graph paper, the largest area that can be made has an area of 48 $cm^2$. This can be found by removing one square out of one corner of a 7 x 7 square. The area of the largest hexagon with a perimeter of 52 cm is 68 $cm^2$, a 13 x 13 square with one square removed from one corner.

Question **6**

Because the perimeter of these hexagons is the same as the perimeter of its related rectangle, a hexagon with a perimeter of 25 cm is not possible on cm graph paper.

Questions **7 - 11**

The hexagon is suggested here instead of a rectangle because it is assumed that children will have had some experience with finding rectangles of different areas that have the same perimeter. That is, children should have some experience investigating questions such as “Can you find two different rectangles that have a perimeter of 20, but that have different areas”? Whether working with the rectangle or with the hexagon, children should note that half the distance around the shape is half of the total perimeter. This is clear for a rectangle, but the same principle holds for the hexagon. Knowing this helps children more readily find hexagon shapes for a given perimeter.

Graph paper is used so that children can more readily focus on area and perimeter rather than on the task of making a shape with a ruler. For a hexagon, the task is harder for students without such guidance, but you might consider not using line paper when investigating similar questions using rectangles.

A calculator is useful for the multiplication and addition used in finding the areas of the two rectangles combined. While most of the computations might be done mentally, calculator use prevents computational difficulties from hindering the problem-solving experience.

Question **12**

The values 28 and 52 were chosen because they are multiples of 4 and can lead to making hexagon shapes that are close to squares.

## *Activity 3: Eggsactly*

Adapted with permission from "An Egg Sample of Birds" from *World's Largest Math Event 6*: *Animals as our Companions*, copyright 2000, by the National Council of Teachers of Mathematics. All rights reserved.

Children are familiar with eggs, whether those you eat, those that are used for decorative purposes, or pictures of bird eggs that can be seen in a book. There are many interesting investigations that can arise from a study of the size of eggs.

Val and Maria were examining eggs when they wondered if there was a relationship between the length and width of an egg. Using calipers to measure the eggs, they gathered the data given below. The table below and the following questions will help you determine whether there is a relationship between the length and width of an egg.

| Bird Name | Egg Length | Egg Width |
|---|---|---|
| Canada Goose | 8.6 cm | 5.8 cm |
| Robin | 1.9 cm | 1.5 cm |
| Turtledove | 3.1 cm | 2.3 cm |
| Hummingbird | 1.0 cm | 1.0 cm |
| Raven | 5.0 cm | 3.3 cm |
| Chicken | ? | ? |

Answer questions 1 - 4 as a *learner* of mathematics.

**1.** What do you predict the length and width of a chicken egg to be? Will it matter if you have a small, medium, large, or extra-large egg? Measure a chicken egg and add your finding to the table.

**2.** Make clay models that approximate the dimensions of the eggs. How much greater is the volume of the Canada goose egg compared to that of the robin? How much greater is the volume of the turtledove egg compared to that of the hummingbird?

**3.** If an ostrich egg was 21 cm long and 15 cm wide, how many hummingbird eggs would it take to equal the volume of the ostrich egg?

**4.** Use a TI-73 to make a scatter plot to find out if a relationship exists between the lengths and widths of bird eggs. If appropriate, use the calculator to produce a regression equation for the relationship between the length and width of the bird eggs.

Answer questions 5 - 9 as a *teacher* of mathematics.

**5.** How accurate was your prediction for the length and width of a chicken egg?

**6.** How can you make use of the weight of the eggs to help answer a question about the relationship of volumes of the eggs? (If the clay model of one egg weighs twice the clay model of another other egg, it will have twice the amount of clay. Hence, it will also have twice the volume.)

**7.** How would you try to get students to understand the reason for the use of clay?

**8.** If you did not use the clay to make comparisons between the volumes of the eggs, what other approach could be used? (Displacement is one method.)

**9.** Do you think that a relationship can be determined between the lengths and widths of potatoes of the same variety?

### *Answers and comments*

Questions **1 – 4**

The dimensions for a chicken egg should be found by obtaining an egg and making the measurements. Finding the length and width of an egg takes some patience. Children should not be allowed to make these measurements along the surface of the egg.

Using a clay model for each of the eggs and using a scale is one of the best ways to determine the relationships between the volumes of the eggs, including the ostrich egg. However, it is important that children do not associate the mass of the clay with the volume of the egg. The clay is being used to model a situation so that a comparison can be made rather than a volume directly determined, although the volume for an egg could be made by the displacement method of submerging the clay model of the egg in water.

Numerous predictions are expected when children try to answer if there is a relationship between length and width. Some will think there is no relationship, some will say that the length is usually longer than the width, while others may respond that the width appears to be about two-thirds of the length. Of course, the hummingbird egg does not work well with any prediction.

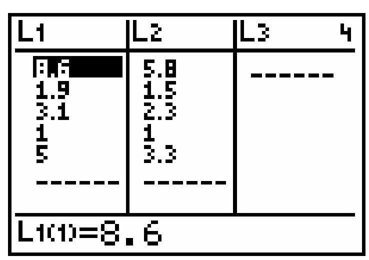
For those for whom it makes sense, a regression line can be found for this example.

After finding the regression line relating the lengths and widths of the eggs, students may wish to check to see how well the predictions hold for various eggs.

To make a scatter plot on the TI-73, follow the steps below.

**1.** Press [LIST] to display the List editor. If there is information in the list, clear it by positioning the cursor on the heading **L1** and then pressing [CLEAR] [ENTER].

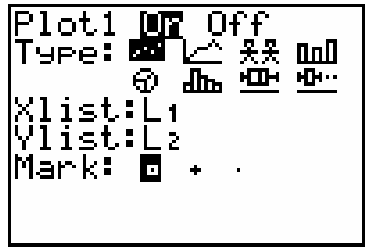
**2.** Enter egg length in **L1**, and egg width in **L2**.




**3.** Press [2nd] [PLOT].

**4.** Press [ENTER] to select **1:Plot1**.

5. Press [▶] to select **On**, and then press [ENTER]. Set your TI-73 as shown at the right, and described below.

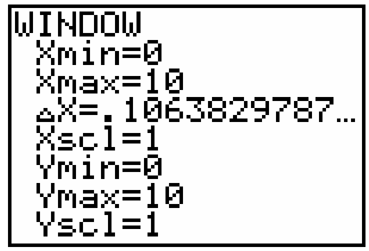



   a. Press [▼] to select **Type.** To set the plot **Type**, press [▶] to highlight (scatter plot), and then press [ENTER].

   b. Press [▼] to select **Xlist:.** Press [2nd] [STAT]. Highlight **1:L1** and then press [ENTER].

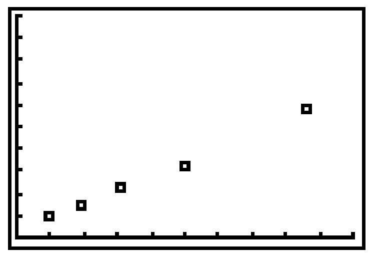
   c. Press [▼] to select **Ylist:.** Press [2nd] [STAT]. Highlight **2:L2** and then press [ENTER].

6. Press [WINDOW]. Make sure the **Xmin**, **Xmax**, **Ymin**, and **Ymax** have values that fit the data from your lists. Using the sample data above, you could set your **WINDOW** as follows:



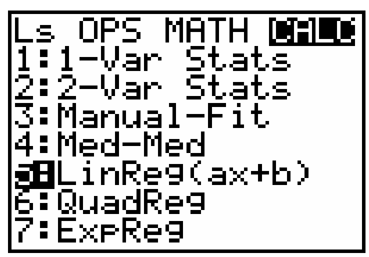

| | |
|---|---|
| **Xmin** = 0 | **Ymin** = 0 |
| **Xmax** = 10 | **Ymax** = 10 |
| **Xscl** = 1 | **Yscl** = 1 |

7. Press [GRAPH] to display the graph.

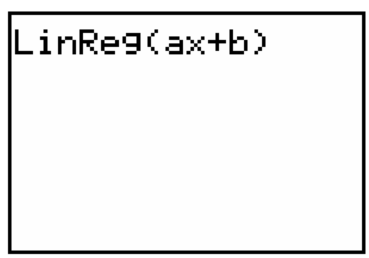


To make a linear regression on the TI-73, follow the steps below.

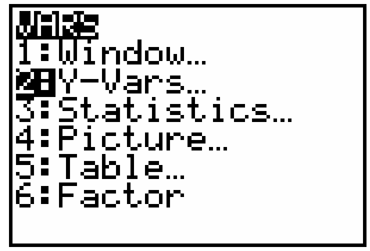
1. Press [2nd] [STAT]. Press [◀] to highlight CALC. Press **5** to choose **5:LineReg(ax+b).**



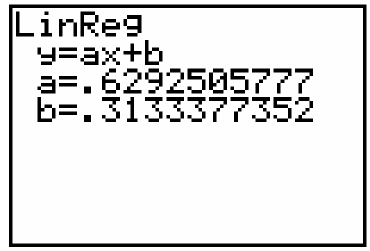

   **LineReg(ax+b)** is displayed on the Home screen.



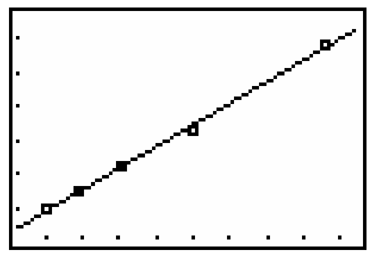
**2.** Press [2nd] [VARS]. Press **2** to choose **2:Y-Vars**.




**3.** Press **2** to choose **2:Y2** and then press [ENTER]. This gives an expression relating the length and width of the eggs. That is, the width = 0.6292505777 x length + 0.31333773532. If you used data from a chicken egg, these values would be different.




**4.** Press [ZOOM]. Press **7** to choose **7:ZoomStat** and view the line on the Graph screen. This shows that all the data points are very close to the line in step 3 above.



Questions **5 – 9**

It may be surprising that the use of clay and the process of weighing can be helpful for this investigation. However, whether comparing the weight of the clay models, or using the clay models to find the volume by displacement, an accurate measurement can be determined.

The prediction of the length and width of a chicken egg would depend upon some knowledge of a chicken egg and the type of egg (small, medium, large, extra-large) considered. Whatever size of egg was chosen should fit relatively nicely into the data set. This activity can be explored with potatoes in the same manner as exploring eggs. Students need to make sure that they define and measure length and width consistently. Displaying the data in a table or as a function on a graphing calculator will work just as in the egg exploration. Students may like to explore measurements of other items to see if linear relationships exist.

## Using calculators to assess geometry and measurement

A calculator can be a useful tool when exploring problems in geometry and measurement. It can be used to simplify calculations so students can gather more data. It can provide graphical interpretations for geometric relationships. Two assessment items for geometry and measurement are provided below.

### Assessment 1: Pacing Meters

**1.** Jud was measuring his stride while walking 10 meters. After a few experiments, he found that he consistently walked 12 steps to cover 10 meters.

- **a.** If he walks 98 steps, how many meters has he walked?
- **b.** While walking 100 meters, how many steps will Jud take?

**2.** Jud's sister Lem walks 10 meters in 15 steps.

- **a.** Who has the longer stride?
- **b.** Who will take the most steps, and how many more, to cover 250 meters?

### Answers and comments

Question **1**

If Jud goes 98 steps, he will go between 81 and 82 meters. Jud will take 120 steps to go 100 meters.

Question **2**

Jud has a longer step than Lem. Jud takes 300 steps to cover 250 meters and Lem takes 375 steps.

### *Assessment 2: Growing Cubes*

1. Jari was experimenting with cube blocks. He started with a single cube and put a circular dot on each of face. He then built a cube two units on a side and put circular dots on each of the squares on each face.

   a. How many circular dots did he use on the first cube?

   b. How many cubes did he need to make the second cube? How many circular dots did he use on the second cube?

2. Jari thought he saw some patterns emerging, so he built cubes with three, four, and five units on a side, and put circular dots on each of the squares on each face of the other cubes.

   a. How many cubes and circular dots were used for each cube he made?

   b. How many cubes would he need for a cube with 20 units on a side?

   c. How many circular dots would be needed if he put a dot on each square of the cube with 20 units on a side?

### *Answers and comments*

Question **1**

One cube will have six dots. A cube with two units on a side will have 8 blocks and 24 dots.

Question **2**

A cube with three units on a side will have 27 cubes and 54 dots. A cube with four units on a side will have 64 cubes and 96 dots. A cube with five units on a side will have 125 cubes and 150 dots.

In general, a cube with $n$ units on a side will have $n^3$ cubes and $6n^2$ dots. Thus, a cube with twenty units on a side will have 8000 cubes and 2400 dots.

## *Activity overviews for K-6: geometry and measurement*

The following list contains brief descriptions of elementary school activities that incorporate the use of the calculator as a recording or exploring device for developing understanding of geometry and measurement. The activities can be found on the CD that accompanies this text.

***Drip, Drip, Drip*** (*Using the TI-73: A Guide for Teachers*, Texas Instruments, 1998.)

Students collect data about the number of drips from a faucet and use a TI-73 to explore the amount of water that is wasted over a specific period of time. The idea of rate of change is explored as well as determining an equation for predicting the amount of water for any elapsed time.

***Only the Height has been Changed*** (*Using the TI-73: A Guide for Teachers*, Texas Instruments, 1998.)

Students collect data and examine the effect the height of a ramp has on the distance a toy car will travel when rolled down the ramp. A TI-73 can be used to provide a visual representation, and if appropriate, generate an equation relating the height of the ramp to the distance the car travels.

***Farmer's Chicken Coop*** (Mankus, Margo and Klespis, Mark, eds. *Geometry and Measurement Module, Math Teacher Educator Short Course for College Professors Teaching Pre-service Teachers with an Elementary School Math Focus*, Technology College Short Course Program, 1999.)

Students are to determine the largest amount of room for chickens given a certain amount of fencing. The Math Explorer™, TI-10, TI-15, or TI-73 can be used with this exploration.

***What is Your Heartbeat*** (Mankus, Margo and Klespis, Mark, eds. *Geometry and Measurement Module, Math Teacher Educator Short Course for College Professors Teaching Pre-service Teachers with an Elementary School Math Focus*, Technology College Short Course Program, 1999.)

Students explore various relationships between their heartbeat and other physical activities. A calculator can be used to provide a statistical picture, to graph relationships, or perform calculations.

***Do Centimeters Make Me Taller?*** (Schielack, Jane and Chancellor, Dinah. *Uncovering Mathematics with Manipulatives and Calculators Levels 2-3*, Texas Instruments, Dallas, TX, 1995.)

Students make comparisons between standard and non-standard units of measurement to measure a variety of objects and use the TI-10, TI-15 or Math Explorer™ to examine ratios between the measurements.

***What's My Ratio?*** (Schielack, Jane and Chancellor, Dinah. *Uncovering Mathematics with Manipulatives and Calculators Levels 2-3*, Texas Instruments, Dallas, TX, 1995.)

Students use the Math Explorer™ to investigate proportionality and determine the constant ratio between similar figures as they are stretched or shrunk.

***Do You See What I See?*** (*Discovering Mathematics with the TI-73: Activities for Grades 5 and 6,* Texas Instruments, 1998.)

Students use the TI-73 to perform coordinate graphing through the use of lists. Students explore how pictures formed can be stretched and shrunk by multiplying and dividing the coordinates by constant values.