Chapter 2

Number and Operation Sense: Whole Numbers

In this chapter

This chapter summarizes NCTM's ***Principles and Standards for School Mathematics*** *for number and operation sense and applies those principles in some activity overviews. You will learn how to:*

- *Use the calculator to develop operation sense in terms of the effect an operation has on the numbers involved.*
- *Use the calculator to encourage exploration of the mathematical principles behind the standard multiplication algorithm for multidigit numbers.*

Overview of number and operation sense: whole numbers

Number sense is the ability to use numbers in describing quantities and in computing. It allows you to decompose numbers into various parts, to use appropriate numbers such as 10 and 100 as referents, to understand the base ten numeration system, to estimate quantities, and to recognize the sizes of numbers and their relationships to other numbers (Sowder, 1992).

Operation sense includes an understanding of the types of questions an operation can answer and the effects an operation has on the numbers involved.

Goals for students

Students in elementary grades begin to attain a basic understanding of numbers that grows as they progress through grade 12.

According to NCTM's *Principles and Standards for School Mathematics*,

> understanding number and operations, developing number sense, and gaining fluency in arithmetic computation form the core of mathematics education for the elementary grades. As they progress from prekindergarten through grade 12, students should attain a rich understanding of numbers—what they are; how they are represented with objects, numerals, or on number lines; how they are related to one another; how numbers are embedded in systems that have structures and properties; and how to use numbers and operations to solve problems. (p. 32)

Young children learn the concept of number from comparing sets of objects and generalizing the idea that is common to the sets. For example, the sets {red, blue, yellow}, {dog, cat, mouse}, and {train, car, bicycle} have as a common idea the number *three*, represented by the numeral **3**. Extensive exploration with concrete models of sets provides the basis for students to develop number sense.

Young children learn about operations on numbers by answering questions such as "How many are there in all?" "How many are left?" "How many are in each group?" and "How many groups are there?" Young children can use calculators to learn the operations and symbolism needed to represent the number sentences that answer these questions, as shown in the sample activities in this chapter. As elementary students explore the operations with the calculator, they develop operation sense. You certainly do not want students to depend upon the calculator to do simple computations. However, in using the calculator to explore numbers and operations and their symbols, students engage in a vast amount of practice and feedback that supports recall of basic facts. As they learn paper-and-pencil and mental algorithms for computing, students can use the calculator as a device for exploring the mathematical principles represented in the algorithms. As students use the calculator for computation with larger numbers, the number and operation sense they have developed in their early explorations provides them with estimation skills that allow them to determine whether their computational results are reasonable or not.

Goals for teachers

According to the Conference Board of the Mathematical Sciences (CBMS), in their document *The Mathematical Education of Teachers,* to be well prepared to teach arithmetic for understanding, elementary teachers need to understand:

- A large repertoire of interpretations of addition, subtraction, multiplication and division, and of ways they can be applied.
- Place value: how place value permits efficient representation of whole numbers and finite decimals; that the value of each place is ten times larger than the value of the next place to the right; implications of this for ordering numbers, estimation, and approximation; the relative magnitude of numbers.
- Multidigit calculations, including standard algorithms, "mental math," and non-standard methods commonly created by students: the reasoning behind the procedures, how the base-10 structure of number is used in these calculations.
- Concepts of integers and rationals: what integers and rationals (represented as fractions and decimals) are; a sense of their relative size; how operations on whole numbers extend to integers and rational numbers; and the behavior of units under the operations (CBMS, 18).

In this chapter you will learn how to use the calculator to develop operation sense in terms of the effect an operation has on the numbers involved. By studying some sample activities that deal with number and operation sense, you will also learn how to encourage exploration of the mathematical principles behind the standard multiplication algorithm for multi-digit numbers.

Sample activities

These activities are designed to help teachers learn the mathematics involved in working with whole numbers and reflect on what they have learned.

Activity 1: 1 – 2 – 3 – 4 Challenge

This activity explores operation sense in terms of the effect an operation has on the numbers involved. It requires the user of the calculator to learn which of two logic structures the calculator uses for computing multiple operations. The two logic structures are arithmetic and algebraic. Arithmetic logic structure computes operations in the order they are entered. Algebraic logic structure computes operations in the algebraic order of operations (parentheses first, then exponentiation, then multiplication, division, addition, and subtraction, in that order).

Use the four digits **1, 2, 3, 4**, and any or all of the four operations: **+, -, x, ÷** to form one or more expressions for each of the values 1 through 12. Test your expressions on the calculator.

Answer questions 1 – 3 as a *learner* of mathematics.

1. Share with a classmate one of the expressions you created. (For example, an expression for 6 is **3 x 4 ÷ 2 x 1**.) Did you both create the same expression for 6? How are your expressions alike or different?

2. Consider the following expression for 12: **3 + 4 x 2 + 1**. Try it on the TI-108 calculator. Try it on the MathMate™, TI-10, or TI-15 calculator. What happens? Why do you think this happens? What happens if you use the expression **4 x 2 + 3 + 1** on the two calculators? Which calculator uses which kind of logic structure?

3. What strategies did you use to create expressions for each value? What properties of the operations were involved in these strategies?

Answer questions 4 – 6 as a *teacher* of mathematics.

4. What important knowledge of characteristics of whole number operations does this activity build?

5. When might you use an activity of this type in the curriculum?

6. One of the disadvantages of using a calculator in an activity is that there is little written record of the students' thinking or actions. How would you encourage students to record their thinking, as well as their results in this activity?

Answers and Comments

Question **1**

It is important to see the variety of expressions that can be created for each value. Many algebraic properties can be explored by comparing two different expressions.

Question **2**

The TI-108 is a calculator that uses arithmetic logic—in other words, it computes in the order in which the operations are entered. So on this type of calculator, the result of entering **3 + 4 x 2 + 1 =** will display as **15**, and the result of entering **4 x 2 + 3 + 1 =** will display as **12**. The MathMate™, TI-10, TI-15, and most other TI calculators use algebraic logic and employ the algebraic order of operations (parentheses first, then exponentiation, then multiplication, division, addition, and subtraction, in that order). Therefore, both **3 + 4 x 2 + 1 =** and **4 x 2 + 3 + 1 =** display as **12**, since the product **4 x 2** is always computed first using the algebraic order of operations. However, using parentheses in the expression **(3 + 4) x 2 + 1 =,** would give a display of **15**.

Question **3**

Whole number operation sense ideas such as multiplying or dividing by 1 to keep the same number, increasing values with addition or multiplication, and decreasing values with subtraction or division are helpful for developing strategies for finding expressions. Also, number theory ideas about even and odd numbers can be useful.

Questions **4 - 6**

To design trials and conjectures in this activity, students have to think about the effects of operations on whole numbers – how do I make something bigger or smaller, or make it stay the same (for example, +0 or x1?) After students have developed an understanding of the meaning of an operation, they can use the calculator to explore its effects using a variety of numbers. However, students must be guided and encouraged, through well-designed recording sheets or class charts, to record and organize the information they create in their explorations in order to identify patterns and make and test predictions. For example, 1 x 2 = 2, 2 x 2 = 4, 3 x 2 = 6, 4 x 2 = 8, and so forth, seems to indicate that n x 2 yields an even number.

Activity 2: Create the Greatest Product

In this activity, you will use the calculator to explore the mathematical principles behind the standard multiplication algorithm for multi-digit numbers. Note that place value and the distributive property of multiplication over addition play significant roles in this exploration.

Answer questions 1 – 4 as a *learner* of mathematics.

1. Using the five digits **1**, **2**, **3**, **4**, and **5**, place one digit in each box in the template to create the factors using these five digits that give the greatest product. Use each digit exactly one time. Record your first, second, and third attempts in the appropriate places below, using your calculator to compute the products. Answer the accompanying questions.

First Attempt

x		

Second Attempt

x		

Third Attempt

x		

- **a.** Why did you place the digits where you did in your first attempt?
- **b.** Why did you place the digits where you did in your second attempt?
- **c.** Did putting the greatest digits in the greatest-place-value positions give you the greatest product?
- **d.** How do you know that you have the greatest product by the third attempt?

2. Try the problem again with the digits **2**, **4**, **6**, **7**, **9**.

First Attempt

x		

Second Attempt

x		

Third Attempt

x		

- **a.** Why did you place the digits where you did in your first attempt this time?
- **b.** Did you need to make a second attempt? Why or why not?
- **c.** What conjecture can you make about placing the digits to create the greatest product?
- **d.** How can you test your conjecture?

3. Try the problem again with the digits **0**, **3**, **5**, **6**, **8**. Compare your results to the conjecture you made in question **2c**. How is this set of digits different from the others? Does it fit the conjecture or not?

4. Use mathematical properties to explain why the pattern in the conjecture works.

Answer questions 5 – 9 as a *teacher* of mathematics.

5. How does this activity reinforce basic multiplication facts?

6. How does this activity encourage development of number and operation sense with whole numbers?

7. When might you use an activity of this type in the curriculum?

8. How would this activity be different without the use of the calculator?

9. How would you guide students from a trial-and-error method of using the calculator to solve the problem to a method based on mathematical principles?

Answers and Comments

Question **1**

Most people, on their first attempt to solve this problem, place the greatest digits in the positions of greatest place value: **532 x 41 = 21,812.** However, when encouraged to try something else, or in comparing it to another person's attempt, they soon find that this is not the greatest product. For example, **531 x 42 = 22,302.** An unexpected finding is that **432 x 51 = 22,032**, which is greater than one of the other products with the **5** in the hundreds place. The greatest product is obtained with the factors **431 x 52 = 22,412.**

Question **2**

If you follow the pattern learned in the previous problem, your first attempt is **762 x 94 = 71,628.** Attempts of other factors shows that this is the greatest product. From these two problems, one can conjecture that the greatest product is formed by placing the greatest digit in the tens place of the two-digit factor, the next greatest digit in the hundreds place of the three-digit factor, the next greatest digit in the tens place of the three-digit factor, the next greatest digit in the ones place of the two-digit factor, and the smallest digit in the ones place of the three-digit factor. If the digits are labeled and ordered such that $a < b < c < d < e$, then the factors that create the greatest product can be represented as *dca* times *eb*.

Question **3**

If the conjecture is tested with this set of digits, **650 x 83 = 53,950** is the greatest product. But with some experimentation, it can be shown that this same product is also obtained with **830 x 65**. Does this discovery disprove the conjecture? No, because the conjecture that *dca* x *eb* produces the greatest product is still true, but is not the *only* set of factors that has the greatest product. Why does *eba* x *dc* work in this situation and not in the others? Could it be because of the **0** digit? Students who test other sets of digits that contain 0 will find two pairs of factors that give the greatest product.

Question **4**

The problem in question 3 can help lead to the understanding of the mathematical principals involved here.

For example, **650 x 83 = (65 x 10) x 83 = 65 x (10 x 83) = 65 x 830**, based on the associative property of multiplication. However, if there are more than zero ones in the ones place, the representation changes.

For example, **651 x 83 = (650 + 1) x 83 = (65 x 10 x 83) + (1 x 83)**, based on the distributive property of multiplication over addition.

However, **831 x 65 = (830 + 1) x 65 = (83 x 10 x 65) + (1 x 65).** Note that the first part of the sum is the same in both products (65 x 10 x 83) = (83 x 10 x 65); however, the second part of the sum is different (1 x 83) ≠ (1 x 65), and its size is determined by the size of the two-digit factor.

Therefore, although **650 x 83 = 830 x 65, 651 x 83 > 831 x 65.**

Questions **5 – 9**

Using a calculator for the actual computation provided (1) enough data in a short amount of time to be able to make and test conjectures, and (2) time and motivation to actually dissect the algorithm (see Question **4**) and search for mathematical principles to support or refute your conjectures. Knowledge of basic facts of multiplication were important and used in estimation and elimination of unreasonable trials, such as **351 x 24**, since **3 x 2 = 6** is obviously too few thousands to be the greatest factor when you could have **5 x 4 = 20**.

Using calculators to assess whole number and operation sense

Using calculators to assess students' understanding of whole numbers and whole number operations requires asking for more than just a computational result. The role of the calculator in the assessment item should be (1) to collect enough data to be able to look for patterns and make conclusions about numbers or operations or (2) to assist in complicated computations in order to provide time for concentrating on the number or operation concept being assessed.

For example, in the first assessment item below, students can use the calculator to test a wide variety of sets of three consecutive whole numbers in order to look for a pattern in their sums. In the second assessment item, students can focus on the meanings of each of the operations in order to select the appropriate ones to answer the questions, and then use the calculator to obtain the exact numerical answers.

Assessment questions

1. For every set of three consecutive whole numbers, such as {3, 4, 5}, what conclusions can you make about their sums?
2. You are planning a cross-country trip from San Diego, California to Caribou, Maine. Locate the data you need to answer the following questions. Decide which operation(s) you need to use, compute the answers, and determine if your answer is reasonable or not. If you obtain an unreasonable answer, review your choices of data and operations and revise your procedure.

   a. How many miles is it from San Diego, California to Caribou, Maine? What processes and operations did you use to determine your answer? Why?

   b. If you can drive 315 miles per day, how many days will it take you to make your trip? What processes and operations did you use to determine your answer? Why? Is 315 miles per day reasonable? Why or why not?

   c. If your car averages 23 miles per gallon on the highway and 18 in the city, how much gasoline do you expect to use? What processes and operations did you use to determine your answer? Why?

   d. If gasoline averages $1.95 per gallon across the country, how much do you expect to spend on gasoline? How much would you save if gasoline prices dropped 8 cents per gallon across the country? What processes and operations did you use to determine your answer? Why?

   e. If this was a business trip, you could get reimbursed 33 cents per mile. How much would your reimbursement be? What processes and operations did you use to determine your answer? Why? Would that be a fair reimbursement for your travel expenses? Why or why not?

Answers and comments

Question **1**

Students usually begin with sets of small whole numbers for which they do not need the calculator (for example, 3 + 4 + 5 = 12 and 10 + 11 + 12 = 33). From these sets, one conclusion they can come to is that each of the sums is divisible by 3. They can then use the calculator to test their conjecture on a wide variety of sets of three consecutive whole numbers (for example, 3,423 + 3,424 + 3,425 = 10, 272, and 10, 272 is divisible by three because the sum of its digits is 12). Of course, students should at some point look at the general representation of the numbers involved—n, $n + 1$, and $n + 2$—to be able to prove why the sum ($n + n + 1 + n + 2 = 3n + 3$) is always divisible by 3.

Question **2**

The exact numerical answers to each of these questions will depend upon the route chosen and the mileage data used by each student. However, the choices of procedures and operations will be fairly consistent for determining the answer to each question. For example, **a** requires addition of distances of legs of the trip; **b** requires division of total miles by 315; **c** could involve a complicated process of computing number of miles in the city and number of miles outside the city and dividing by each mpg or determining some process for finding average mpg for the entire trip and using that as the divisor of the total mileage; **d** requires multiplication of gallons of gas needed by the cost and subtraction to compare the two average costs; and **e** requires multiplication of mileage by the 33 cent reimbursement and subtraction to compare the reimbursement to the actual travel expense, which should include deterioration on the car tires, use of oil, and so on. The student then can use the chosen operations and equations to calculate the answers with the calculator. If some answer obtained is unreasonable, the student should be encouraged to review the operation and equation used to represent the situation in order to diagnose any misunderstandings they might have of the operations involved.

Activity overviews for K-6: whole numbers and operations

The following list contains brief descriptions of elementary school activities that use the calculator as a recording or exploring device for developing understanding of whole numbers and whole number operations. The activities can be found on the CD that accompanies this text.

Patterns in Counting (Schielack, Jane F., and Chancellor, Dinah. *Uncovering Mathematics with Manipulatives and Calculators, Level 1*, pp. 6-8. Texas Instruments, 1995.)

Students use the repeating function feature, CONS feature, or OP feature of the TI-108, MathMate™, Math Explorer™, TI-10, TI-15, or TI-73 to count sets of concrete objects, connect number symbols to quantities, and look for place value patterns in the number symbols.

What's the Problem? (*TI-15: A Guide for Teachers*, pp. 34-37. Texas Instruments, 2000.)

Students use the Problem Solving mode on the TI-15 calculator to connect a given number sentence (such as 3 + 2 = ? or 5 - ? = 3) to an appropriate action in a problem situation and use addition, subtraction, multiplication, and division to solve the problem.

Ten, Ten, Ten, Ten . . . (*Math Teacher Education Short Course – Elementary: Number Sense,* pp. 4-5)

Students use the repeating function feature, CONS feature, or OP feature of the TI-108, MathMate™, Math Explorer™, TI-10, TI-15, or TI-73 to discover a pattern for adding ten repeatedly to a given number in order to mentally arrive at the sum of two numbers when one of the addends is a multiple of 10.

a-MAZE-ing Secret Paths (Schielack, Jane F., and Chancellor, Dinah. *Uncovering Mathematics with Manipulatives and Calculators, Level 1*, pp. 14-17. Texas Instruments, 1995.)

Students investigate patterns in place value and the effects of adding and subtracting by using the TI-108, MathMate™, Math Explorer™, TI-10, TI-15, or TI-73 to explore relationships between numbers on the hundred chart.

100 or Bust (Schielack, Jane F., and Chancellor, Dinah. *Uncovering Mathematics with Manipulatives and Calculators, Levels 2-3*, pp. 2-5. Texas Instruments, 1995.)

Students use the TI-108, MathMate™, Math Explorer™, TI-10, TI-15, or TI-73 and place-value materials to explore and connect different models that represent the number game they are playing; compare the models; and use the models to create, analyze and revise their strategies for winning.

Multiplication as Repeated Addition (*Math Teacher Education Short Course – Elementary: Number Sense*, pp. 17-19)

Students use the repeating function feature, CONS feature, or OP feature of the TI-108, MathMate™, Math Explorer™, TI-10, TI-15, or TI-73 to explore one of the definitions of multiplication of whole numbers as repeated addition.

Division as Repeated Subtraction (*Math Teacher Education Short Course – Elementary: Number Sense*, pp. 25-27)

Students use the repeating function feature, CONS feature, or OP feature of the TI-108, MathMate™, Math Explorer™, TI-10, TI-15, or TI-73 to explore one of the definitions of division of whole numbers as repeated subtraction.

Divisibility Rules (Williams, S. E., and Bright, G. W. *Investigating Mathematics with Calculators in the Middle Grades: Activities with the Math Explorer™ and Explore Plus™*, pp. 1-8. Texas Instruments, 1998) and also on the Texas Instruments website at **education.ti.com**.

Students use the Math Explorer™, TI-10, TI-15, or TI-73 to collect data and test conjectures that lead to strengthening their knowledge and understanding of divisibility rules.

Aim High, Aim Low (Lawrence, A., and Wyatt, K. W. *Math Investigations with the TI-30X IIS and TI-34II: Activities for the Middle Grades*, pp. 1-6. Texas Instruments, 1999.)

Students use the TI-108, MathMate™, Math Explorer™, TI-10, TI-15, or TI-73 to explore patterns related to place value in calculations by finding the largest and smallest possible solutions for given digits and operations.