NUMB3RS Activity: Sinks & Sources
Episode: “Hot Shot”

Topic: Directed graphs **Grade Level:** 9 - 12

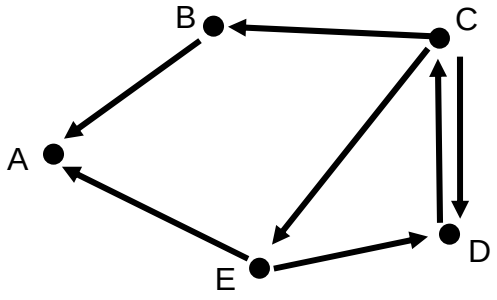
Objective: Investigate paths and flow in directed graphs

Time: 30 minutes

Introduction

In “Hot Shot,” the FBI is looking for a serial criminal. Charlie suggests that by looking at the victims’ daily routines, he may be able to determine where the killer made contact with the victims. He suggests analyzing the information as a directed network flow problem, a mathematical topic that uses directed graphs.

A directed graph can be represented with vertices (points) and edges (arrows) connecting two vertices. Each edge has a direction indicated by the arrow. An example is shown below.



Every vertex has an **indegree** (the number of edges with arrows leading into a vertex) and an **outdegree** (the number of edges with arrows pointing away from a vertex). For example vertex E has indegree of 1 and outdegree of 2. Suppose this graph represents the movements of three victims on a given day. Movements of the victims in a single day include the following: Victim 1 goes from shop C to shop B, and then to shop A; Victim 2 goes from shop C to shop E, and then to shop A; Victim 3 goes from shop E to shop D to shop C, and then back to shop D as shown.

In graph theory Euler paths are routes in which every edge is traveled once. This activity considers Euler paths for which the start vertex and the end vertex are different. (Note: Euler paths and other graph theory topics have been explored in other *NUMB3RS* activities, which are listed on the Extensions page.)

A directed network flow problem assigns a value to each edge of a directed graph. Much like the Euler paths, the amount of flow into a vertex must equal the amount of flow out, with two exceptions: a **source** (a vertex with only outgoing flow) and a **sink** (a vertex with only incoming flow). This setup can represent situations such as traffic flow, movement in a pipe system, or an electrical current.

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Discuss with Students

1. Determine the indegree and outdegree for each vertex in the sample graph.
2. Is there an Euler path in the sample graph using the directed arrows in the directions given?
3. Is there an Euler path if the directions of the arrows are not used? If so, what is it?

Discuss with Students Answers:
1.

Node	*Indegree*	*Outdegree*
A	*2*	*0*
B	*1*	*1*
C	*1*	*3*
D	*2*	*1*
E	*1*	*2*

2. *There is no Euler path that follows the directions of the arrows.* **3.** *There is an Euler path but it does not follow the directions of the arrows. For example, E-A-B-C-E-D-C-D.*

Student Page Answers:
1a. *indegree=7* **1b.** *outdegree=7* **1c.** *These values are equal for any directed graph; the value also equals the number of edges, because indegree counts the number of arrow heads and outdegree counts the number of arrow tails.* **2.** *Graphs 1 and 4 contain Euler paths, Graphs 2 and 3 do not.* **3.** *A directed graph must have one node with outdegree = indegree + 1 (this is the start), one node with outdegree = indegree – 1 (this is the end), and for all other nodes indegree = outdegree* **4a.** Vertices *A and B are sources; the refrigerator and the pantry are where the food is stored, and where it moves from.* **4b.** Vertex *D is a sink; this is a model in which all food eventually goes to the trashcan. It is unrealistic in most circumstances to have that happen.* **4c.** *Normally, not all of the food goes from the table to the trashcan, because most of it is eaten. The leftover food could also be returned to the refrigerator.* **5.** *The table is being used less; maybe it is being used for something else; food is served at the counter.*
6. *Answers will vary, but the outdegree should be greater than the indegree and there should be a path back to both the pantry and the refrigerator.*

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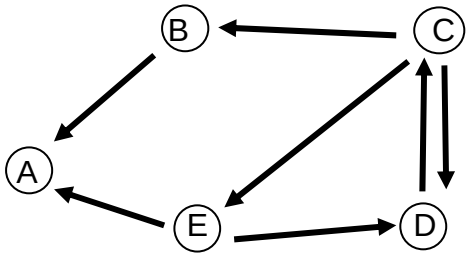
Name _____ Date _____

NUMB3RS Activity: Sinks & Sources

In “Hot Shot”, the FBI is looking for a serial criminal. Charlie suggests that by looking at the victims’ daily routines, he may be able to determine where the killer made contact with the victims. He suggests analyzing the information as a directed network flow problem, a mathematical topic that uses directed graphs.

A directed graph can be represented with vertices (points) and edges (arrows) connecting two vertices. Each edge has a direction indicated by the arrow. Suppose the graph below represents the movements of three victims on a given day. Movements of the victims include the following: Victim 1 goes from shop C to shop B, and then to shop A; Victim 2 goes from shop C to shop E, and then to shop A; Victim 3 goes from shop E to shop D to shop C, and back to shop D as shown.



Every vertex of a graph has an **indegree** (the number of edges with arrows leading into a vertex) and an **outdegree** (the number of edges with arrows pointing away from a vertex).

1. a. What is the sum of the indegrees for the graph above?

b. The sum of the outdegrees?

c. Is it a coincidence that they are equal?

An **Euler path** is a route that covers every edge exactly once. In a directed graph, edges can only be traveled in the indicated direction. This activity considers Euler paths that start and end at different vertices. There are several reasons why the graph above does not contain such a path. Consider vertex A: a path cannot start at vertex A because there are no outgoing edges. When one of the edges going into A is traveled, there is no new edge to take and the path must end.

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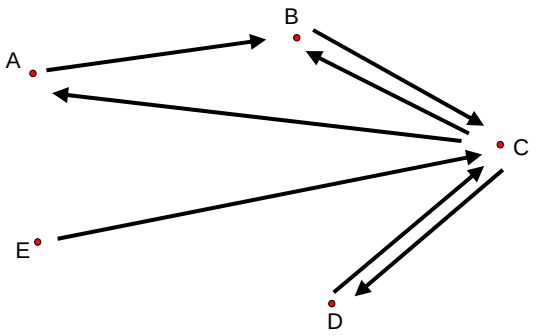
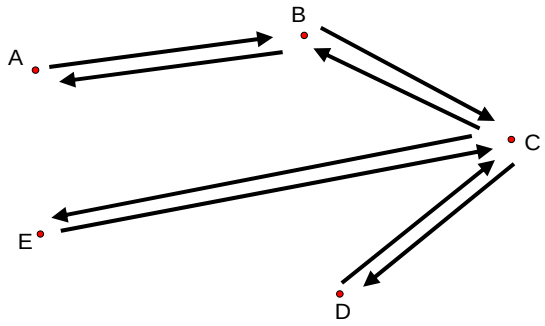
2. Determine whether each graph below contains an Euler path such that the start vertex and end vertex are different.

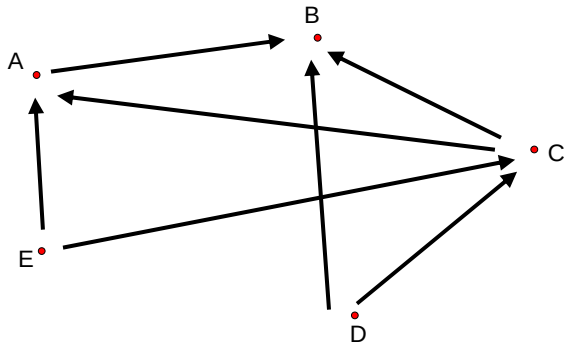
Graph 1



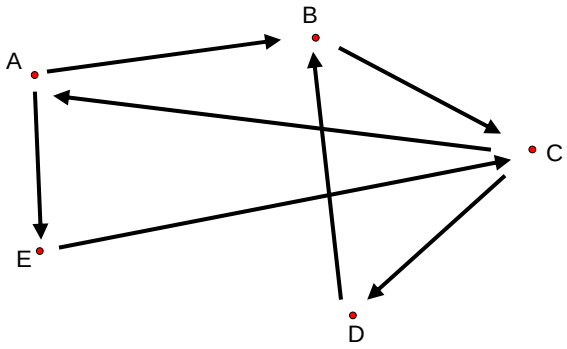
Graph 2



Graph 3

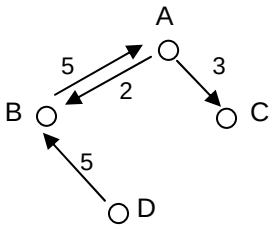


Graph 4



3. Determine the indegree and outdegree for each vertex of the graphs in Question 2. Based on your findings, make a conjecture about Euler paths and directed graphs.

A directed network flow problem assigns a value to each edge of a directed graph. Because these graphs model the flow of material within a system, the amount of flow into a vertex must equal the amount of flow out, with two exceptions: a **source** (a vertex with only outgoing flow) and a **sink** (a vertex with only incoming flow). An example is shown below. Notice that the flow into vertex A is 5 (from vertex B) and the flow out is also 5 (2 to B and 3 to C). Vertex D is a source, only sending material and vertex C is a sink, only receiving material.



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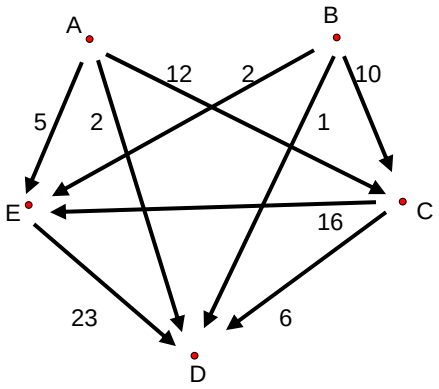
In the *NUMB3RS* episode, Charlie compares the victims’ routines to movements in a kitchen between the table, kitchen sink, refrigerator, and so on. Consider the possible flow of food in a kitchen with vertices representing refrigerator (A), pantry (B), counter (C), trashcan (D), and table (E). The flow value on each edge is a measure of the amount of material transported between locations.

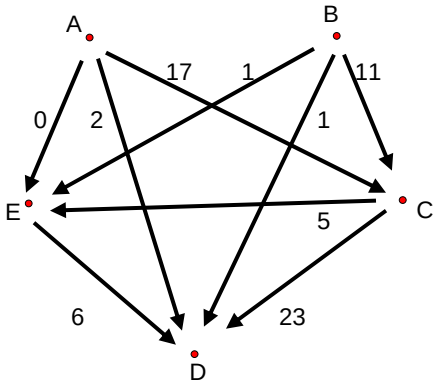


4. a. Which two vertices above are sources? Why does it make sense in this situation that those locations be the sources?

b. Which vertex is a sink? Does this make sense for the situation? Why?

c. In a normal situation, would you expect all the food to go from vertex E to vertex D? Why or why not?

5. Charlie could look for changes in routine as a clue to finding the killer. Look at the revised kitchen flow graph and make a conjecture about how the routine in the kitchen has changed.



6. In reality, the directed flow graph in Question 5 is not realistic. The indegree and outdegree would not be equal in a kitchen because some of the food is eaten. Also, the refrigerator and pantry would not be strict sources because some of the food is returned to both of them. Modify the graph from Question 5 to more realistically model what takes place in a kitchen.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

- Three aspects of directed graphs (or digraphs) are components, cuts, and connectivity. Research these ideas, starting at **http://en.wikipedia.org/wiki/Connected_graph**

- Describe the activity in your school building between classes or after school as a directed graph. Can you use your results to make a proposal to the administration about easing any congestion in the hallways?

Other Resources

- There are additional NUMB3RS activities that cover other graph theory topics. These activities can be downloaded for free from the Web sites below.
  http://www.cbs.com/primetime/numb3rs/ti/activities/Act3_koenigsburgbridge_Toxin_final.pdf
  http://www.cbs.com/primetime/numb3rs/ti/activities/Act1_PartyOfSix_Protest_final.pdf

- This Web site includes more information on graph theory, including directed graphs.
  http://en.wikipedia.org/wiki/Graph_theory

- The idea for using this topic in this episode originated from an interview of Donald Knuth on NPR. Knuth designed his entire kitchen dictated by a graph theory analysis with the central point being the trashcan. You can listen to this interview on the Web site below.
  http://www.npr.org/templates/story/story.php?storyId=4532247

- This Web site, from High School Operations Research, features an activity in which students use graph theory to find the shortest route for a delivery truck.
  http://www.hsor.org/modules.cfm?name=Speedy_Delivery

- This Web site, from the University of Colorado at Boulder, provides activities and information on the use of discrete mathematics, including graph theory.
  http://www.colorado.edu/education/DMP/activities/index.html

- For more work on graph theory, this publication contains an introduction to a variety of topics, including directed graphs.
  Montana Council of Teachers of Mathematics (2003). "Our Town" in *SIMMS Integrated Mathematics: A Modeling Approach Using Technology Level 3*. Dubuque, IA: Kendall/Hunt Publishing Company, pp, 249–272.

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