Mathematical Methods (CAS) Examination 1 Part 1 2003 sample solutions

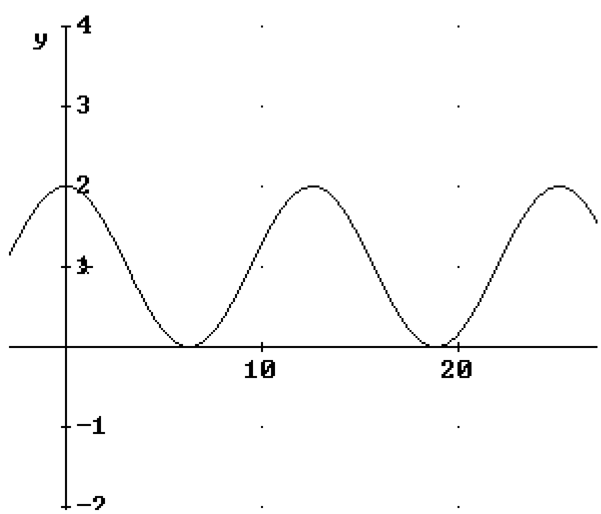
Note: To use *Derive* efficiently, students should be familiar with the 'tick plus equals' and 'tick plus approximately equals' evaluation buttons. These simultaneously 'author' and 'evaluate' expressions exactly and numerically respectively.For example, the 'tick plus equals' or 'author and exact evaluation' button works well for Question 14, while the 'tick plus approximately equals' or 'author and numerical approximation' works well for Question 25. Students should also be familiar with the use of defined functions in th4 form f(x):= *rule of the function*, such as in the sample solutions for Question 4.

Some questions are conceptual in nature, that is, technology will not be of assistance, for example, Question 7. For other questions, such as Question 1, the facility of *Derive* to quickly produce scaled graphs, and the like, means that such problems could be tackled by inspection of each alternative, although this is not a recommended approach.In many cases students will reason what is likely to be the answer, and then confirm this with *Derive*.

Question 1

As no horizontal translation is involved, the graph is a transformed cos graph. The period is 4π, so the graph is an A cos(x/2)+ B graph. The graph has been translated vertically up 1 unit so B = 1, and the graph is a 'right way up' cos graph so A = 1, hence the correct answer is E. Check by drawing the graph, 8π is about 24 units and a bit and 2π is about 6 units and a bit:

#1: $1 + \text{COS}\left(\frac{x}{2}\right)$

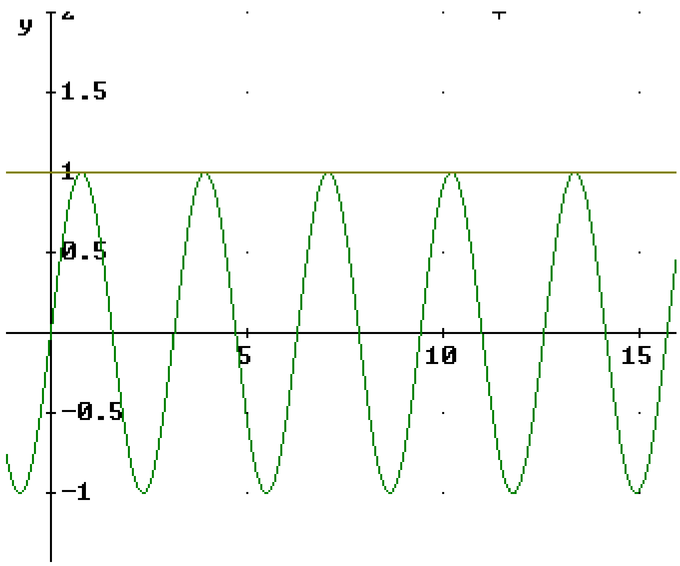


Question 2

This question can be answered without computation. The graph of sin (2x) is the graph of sin(x) dilated horizontally bay a factor of 1/2. Thus while sin(x) has maxima of 1 at π/2, 2π+π/2 = 5π/2, 4π+π/2 = 9π/2 etc, sin(2x) will have maxima of 1 at π/4, 5π/4, 9π/4 etc. Neither A, B nor C provide values sufficently large for the sum of solutions from 0 to 4π, and C does not define a unique exact value (which must exist), hence the answer is E. Given that 4π is larger than 12, this reasoning can be checked visually:

#2: SIN(2·x)

#3: 1



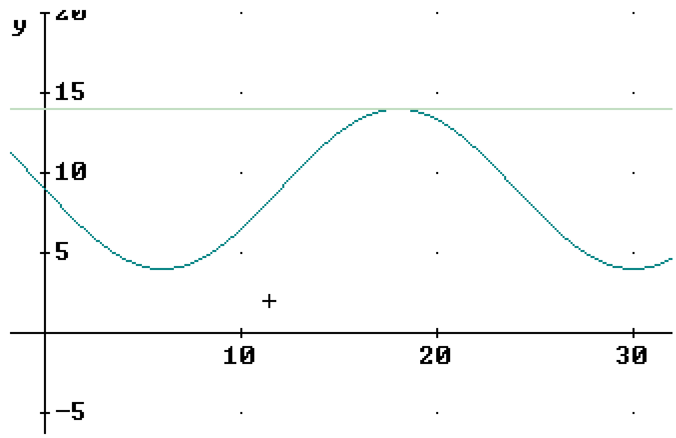
Question 3

This question is best tackled graphically. The graph of y = 14 meets the graph of y = 9−5sin(πt/12) near 18 = 6 pm (best of the available alternatives.

It can also be tackled conceptually.The model has a period of 24 hours. The graph of y = 9−5sin(πt/12) will have its maximim value when sin(πt/12) has the value −1. This will occur when πt/12 = 3π/2 or t = 18 hours = 6 pm.This can readily be checked from the graph:

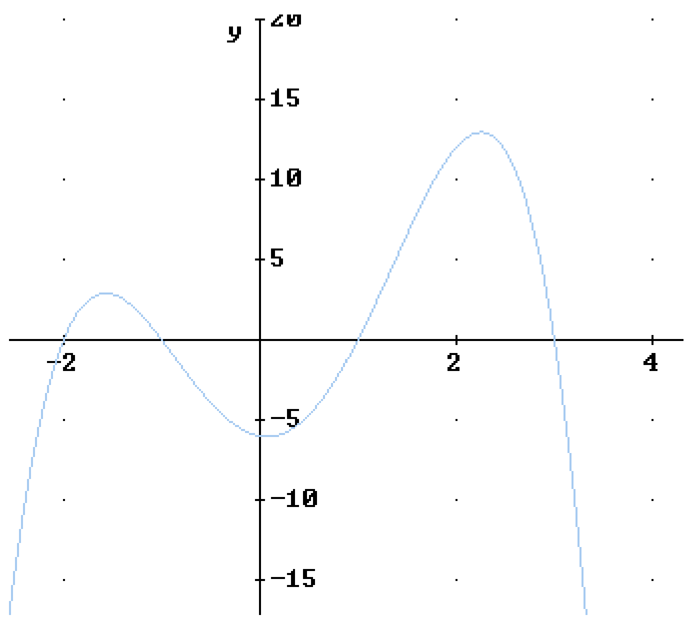
#4: $9 - 5\cdot \text{SIN}\left(\frac{\pi\cdot t}{12}\right)$

#5: 14



Question 4

This question could be done quickly by checking each alternative, however, from the sghape of the graph it is clealry a negative quartic with factors corresponding to roots from left to right of (x+2), (x+1), (x−1) and (x−3). Thus, the required form would be y = −(x+2)(x+1)(x−1)(x−3). There is no alternative directly in this form, however B and C can be eliminated since they represent prositive quartics. A has an incorrect factor (x−2) so either D or E is correct. the (3−x) factor incorporates a negative coefficient, that is (3−x) = −(x−3), so E is correct. D has a 'double negative' and represents a positive quartic. The answer can be chacked graphically:



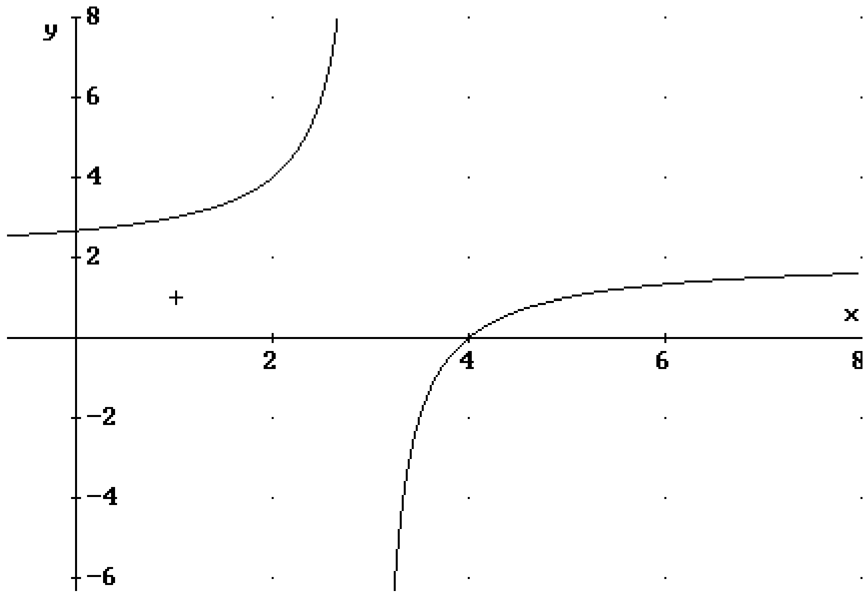
#6: $(x + 2)\cdot(x + 1)\cdot(x - 1)\cdot(3 - x)$

Question 5

A vertical asymptote at x = 3 gives b = −3, so B and C are the only possible correct alternatives. The shape of the graph indicates a 'negative' hyperbola graph, so C can be selected as the correct answer, since the coefficient of a in B is positive.

The answer can be checked by plotting y = −2/(x−3) + 2:

#7: $-\dfrac{2}{x-3}+2$



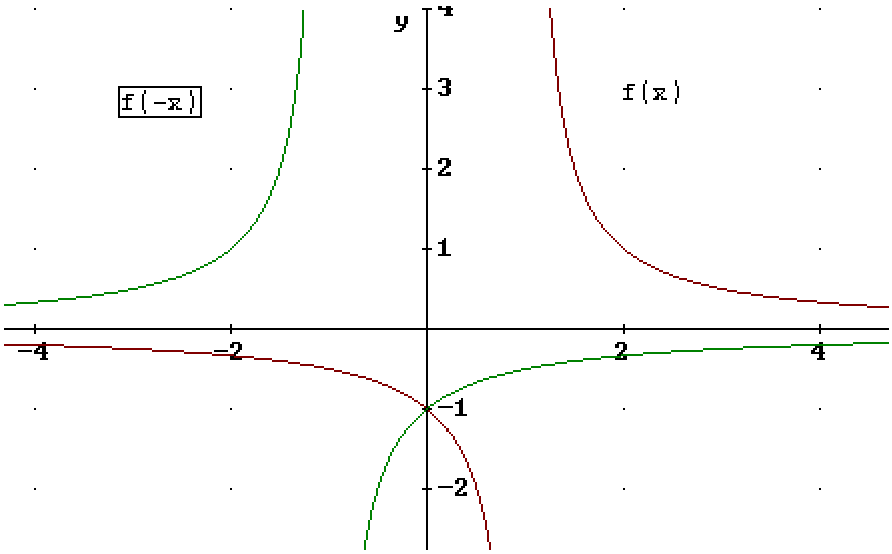
Question 6

This question is a concpetual one, students need to recognise that the graph of y = f(−x) – and its key features such as the vertical asymptote – will correspond to the graph of y =f(x) – and its vertical asymptote – reflected in the vertical, or y, axis. Thus, A is the corrcet alternative.

An alternative, but cumbersome and less efficient approach (and one which is not aleways possible) would be to attempt to model the curve by a specific defined function rule y = f(x) and then observe the graph of f(−x) and then select the best corresponding alternative. in this case care needs to be taken to focus on the relevant features of the graphs.

#8: $f(x) :\!= \dfrac{1}{x-1}$

#9: f(−x)



Question 7

This is a purely conceptual question,students needs to recognise that the graph of the inverse function is obtained by reflection in the line y = x (which they may wish to draw in on the paper to assist themselves), **noting that the diagrams are drawn to a 1 to 1 scale** (even though the scale has not been specified). The correct alternative A can, in this case, be identified by consideration of the linear segment alone.

Question 8

Alternatives D and E can be eliminated readily be checking that there is not a constant firts or second difference respectively. Alternative B can also be eliminated as the log function should have a decreasing rate of increase, whereas this data has an increasing rate of increase. The closeness of the remaining two fucntions can be determined by inspection of the corresponding tables of values for x = 0 to 5 (evaluated using approximate mode):

#10: $\text{TAN}\left(\frac{x}{2}\right)$

#11: $\text{TABLE}\left(\text{TAN}\left(\frac{x}{2}\right), x, 0, 5, 1\right)$

#12:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0.5463024898 \\ 2 & 1.557407724 \\ 3 & 14.10141994 \\ 4 & -2.185039863 \\ 5 & -0.7470222972 \end{bmatrix}$$

#13: $e^{x/2}$

#14: TABLE($e^{x/2}$, x, 0, 5, 1)

#15:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1.64872127 \\ 2 & 2.718281828 \\ 3 & 4.48168907 \\ 4 & 7.389056098 \\ 5 & 12.18249396 \end{bmatrix}$$

Clearly the best alternative is C. For this question, *Derive* could be used to very quickly check the table of values for each alternative.For each of the other rules these are:

#16: $\mathrm{LOG}\left(\frac{x}{2}\right)$

#17: $\mathrm{TABLE}\left(\mathrm{LOG}\left(\frac{x}{2}\right), x, 0, 5, 1\right)$

#18:

$$\begin{bmatrix} 0 & -\infty \\ 1 & -0.6931471805 \\ 2 & 0 \\ 3 & 0.4054651081 \\ 4 & 0.6931471805 \\ 5 & 0.9162907318 \end{bmatrix}$$

#19: $\frac{x}{2} + 1$

#20: $\text{TABLE}\left(\frac{x}{2} + 1, x, 0, 5, 1\right)$

#21: $$\begin{bmatrix} 0 & 1 \\ 1 & 1.5 \\ 2 & 2 \\ 3 & 2.5 \\ 4 & 3 \\ 5 & 3.5 \end{bmatrix}$$

#22: $\frac{x^2}{2}$

#23: $\text{TABLE}\left(\frac{x^2}{2}, x, 0, 5, 1\right)$

#24: $$\begin{bmatrix} 0 & 0 \\ 1 & 0.5 \\ 2 & 2 \\ 3 & 4.5 \\ 4 & 8 \\ 5 & 12.5 \end{bmatrix}$$

Question 9

This is readily done, but students need to note that *linear* factors over *R* are sought, and select the correct field accordingly:

#25: $x^4 + x^3 - 3\cdot x^2 - 3\cdot x$

#26: $$x\cdot(x + 1)\cdot(x + \sqrt{3})\cdot(x - \sqrt{3})$$

thus, B is the correct alternative.

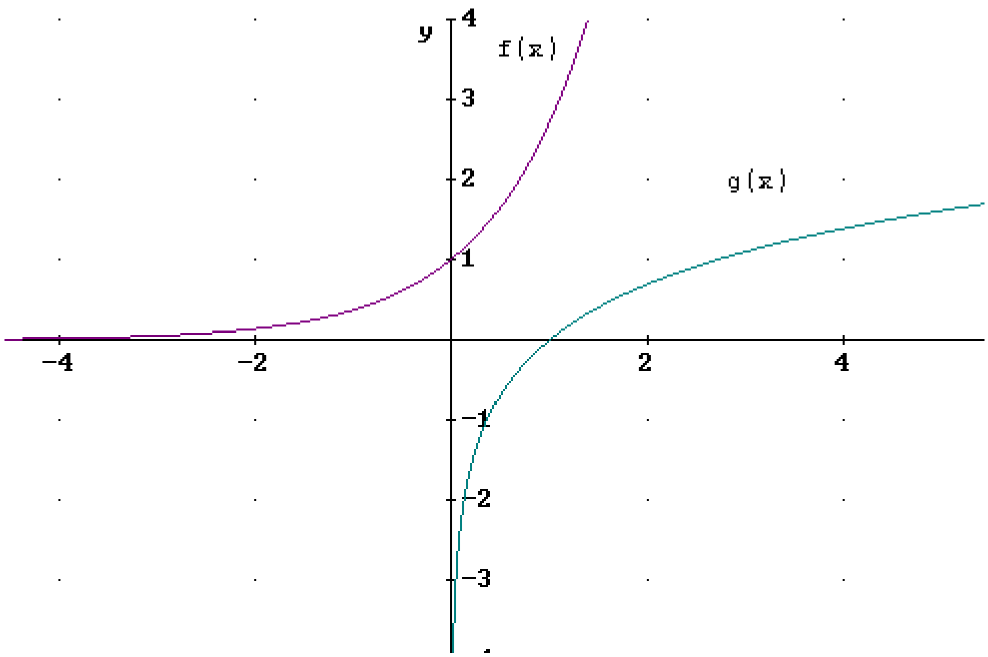
Question 10

This question can be answered directly by noting that the question asks 'which *one* of the following statements is true' and recognising C as a correct statement, from the definition of inverse functions.

If the function rules are defined, each alternative can be tested in turn, although this is much less efficient, and results require interpretation:

#27: $f(x) := e^x$

#28: $g(x) := LOG(x)$



these graphs can be used to eliminate alternatives A and B.

#29: $f(g(x))$

#30: x

this evaluation confirms the correctness of alternative C. No further work is required to answer the question, however the following evaluations are provided to illustrate how each of the other alternatives could be eliminated:

#31: $f(x)\cdot g(x)$

#32: $e^x \cdot LN(x)$

students should be able to see that if x = 1, for example, the product will be e^1*0 which is 0, and thus not equal to 1, hence alternative D is not correct. To eliminate E, evaluate g(f(x)) to see it is not the same as 1/x (this should be known from the definition of inverse functions anyway):

#33: g(f(x))

#34: x

Question 11

This should be known from familioarity with the functions and their graphs, D is the only function which is m - 1 over its natural (implied or maximal domain). This can be tackled graphically, but such an approach should not be necessary in this case.

Question 12

Students should know that for f to have an inverse function, the set A must be a subset of R such that the function f is one to one over A. Thus, any interval which is a subset of the interval from (and possibly including)the x value of the vertex (x = 3) to its left or from (and possibly including)the x value of the vertex (x = 3)to its right will do. D is the only correct alternative. Each alternative could be checked visually against the graph of f(x) over a suitably large domain (note *Derive* automatically expands the rule):

#35: $(x - 3)^2 + 2$

#36: $x^2 - 6\cdot x + 11$