Grade level: 9-12
Subject: mathematics
Time required: 45 to 90 minutes

Cell Phone Towers.

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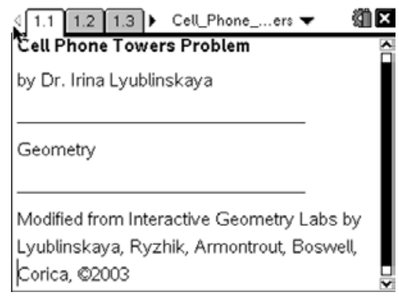
Activity overview



In this activity students explore the locus of a point that is located twice as far from a given point A as it is from given point B. The locus is Apollonius circle. Students discover that the locus is a circle and then prove it. Teacher is encouraged to have students work through the following stages of this problem.

- visualize the locus in your mind without the aid of any materials
- describe or draw a diagram of what has been visualized; the diagram should not be a sketch of the situation
- investigate the problem with TI-Nspire technology; by this stage students should start to consider how the proof of their conjecture will be approached
- give a convincing argument for the solution to the problem

Statement of the problem: *A wireless telephone company has two towers, A and B, for picking up cell phone calls. Tower A is more powerful than tower B. At a location that is twice as far from A as from B, the signals from the two towers have the same strength. When a call is made the tower that has the strongest signal at the phone's location should pick it up. What is the shape of the borderline between the regions where calls should be picked up by tower A and tower B?*

Concepts

If a ray bisects an angle of a triangle, then it divides the opposite side into segments whose lengths are proportional to the lengths of the other two sides. If a ray bisects an external angle of a triangle, then it divides the extension of the opposite side into segments the same way. Consequence: Apollonius circles.

Teacher preparation

Before carrying out this activity teacher should review with the students the concepts of angular bisector, internal and external angles of polygons, and the equation of a circle. The screenshots on pages 2-5 demonstrate expected student results. Refer to the screenshots on page 6 for a preview of the student TI-Nspire document (.tns file).

Classroom management tips

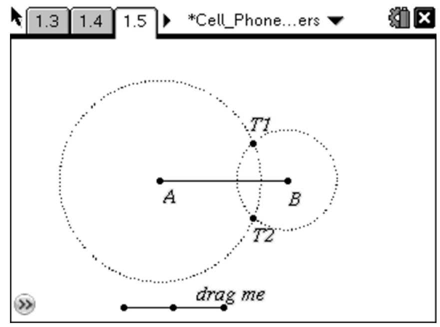
- *This activity is designed to be* ***teacher-led*** *with students following along on their handhelds. You may use the following pages to present the material to the class and encourage discussion. Note that the majority of the ideas and concepts are presented only in* ***this*** *document, so you should make sure to cover all the material necessary for students to comprehend the concepts.*
- *The ideas contained in the following pages are intended to provide a framework as to how the activity will progress. Suggestions are also provided to help ensure that the objectives for this activity are met.*
- *Students may answer the questions posed in the .tns file using the Notes application or on a separate sheet of paper.*
- *In some cases, these instructions are specific to those students using TI-Nspire handheld devices, but the activity can easily be done using TI-Nspire computer software.*

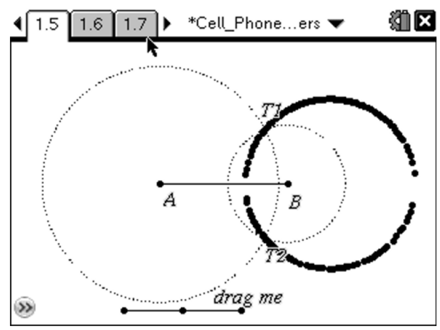
TI-Nspire Applications

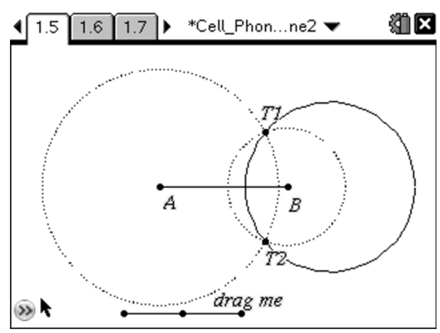
Graphs & Geometry, Notes.

Step-by-step directions

Problem 1 – Locus of Points T, TA:TB = 2:1.







Step 1. Students open file Cell_Phone_Towers.tns, read problem statement on pages 1.2 – 1.3, and move to page 1.4 where they need to answer first question.

Q1. What is the locus of all points T that satisfy this condition? Record you answer and explore on the next page.

A. The correct answer is: the locus of points T is a circle.

At this initial stage of the problem many students may offer incorrect answers. They should not be corrected, since they can correct themselves after the exploration using TI-Nspire.

Step 2. Students should move to page 1.5 to explore this question. On the page 1.5 given fixed segment AB and points T_1 and T_2 that satisfy the condition TA:TB = 2:1. Students can use the slider to vary location of these points and observe their motion. They can choose geometry trace option by clicking (menu) → 5:Trace to observe the path that points T_1 and T_2 follow.

Q2. Construct locus of points T_1 and T_2 on page 1.5. Formulate your conjecture now.

Step 3. In order to construct the locus, students should choose (menu)→9:Construct→6:Locus. Click on the point T_1 and the endpoint of the slider and the locus of T_1 will be displayed as an upper semicircle.

Step 4. Repeat step 3 for the point T_2 and the lower semicircle will be constructed.

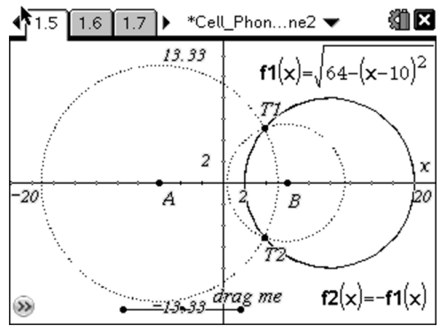
A. The final conjecture is that the locus of the points that satisfy given condition is a circle with the diameter KL, such that points K and L belong to the line AB and KA: KB = LA: LB = 2:1. This circle is called circle of Apollonius.

Q3. Find loci of points T1 and T2, find an equation for the locus and then graph it.

A. It is expected that all students will be able to determine the center of the circle and its radius and come up with two functions, one for the upper circle and one for the lower circle. For this problem, A(-6,0) and B(6,0). Then endpoints of the diameter of the circle are (2,0) and (18,0). Thus the center of the circle is a midpoint of this segment so it is located at (10,0). The radius of the circle is 8. Thus equation of the circle is $(x-10)^2 + y^2 = 64$.

Corresponding equations are $f_1(x) = \sqrt{64-(x-10)^2}$ and $f_2(x) = -f_1(x)$

Step 5. Students can show entry line by pressing ctrl G and enter these two functions. They can select menu, 2:View, 4: Show axes. The graphs of the functions will be displayed over the locus, if found correctly.



Q4. Prove your conjecture.

A. The final conjecture that students formulate should include all the following statements: Let point A has coordinates $(-a,0)$ and point B has coordinates $(a,0)$. Then the locus is a circle with the center at point $(5/3)a$ and radius $(4/3)a$.

The possible geometric proof for this statement is following:

Let T be such a point that TA: TB = 2:1. Construct segments TA and TB. Construct TK as an angle bisector of the angle ∠ATB and TL as an angle bisector of the complementary angle ∠CTB. Points K and L are on the line AB. By properties of the bisector of the internal angle of the triangle, KA: KB = TA: TB = 2:1. By properties of the angle bisector of the external angle of the triangle, LA: LB = TA: TB = 2:1. Then, points K and L have fixed positions on the line AB.

Since angles ∠ATB and ∠CTB are complementary, their bisectors are perpendicular to each other, so ∠KTL = 90°. Then point T is on the circle with diameter KL. Then, if TA: TB = 2:1, then T is on the circle with diameter KL, where K and L are on the line AB and KA: KB = LA: LB = 2:1. Now let us choose point Y on the circle with diameter KL and show that for all such points, YA: YB = 2:1. Suppose that YA: YB = $t \neq$ 2:1. Then, Y belongs to the circle with the diameter K_1L_1 and K_1 and L_1 are on the line AB with the condition K_1A: K_1B = L_1A: L_1B = $t \neq$ 2:1. Since we proved above that K and L have fixed positions, it means that $K_1 \neq K$ and $L_1 \neq L$, so diameters of the circles are different and circles are different. Then Y does not belong to the original circle, which is contradictory to our assumption.

The possible algebraic proof for this problem is following:

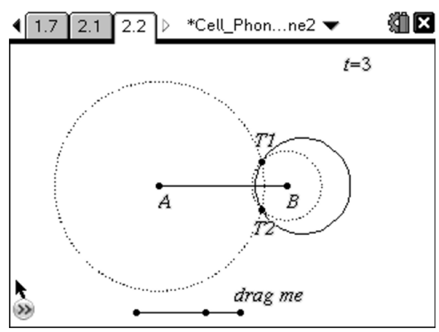
We will solve this problem using coordinate method. Let origin be at the midpoint of the segment AB and coordinates of point A are (-*a*, 0) and coordinates of point B are (*a*, 0). Let coordinates of T are (*x*, *y*).

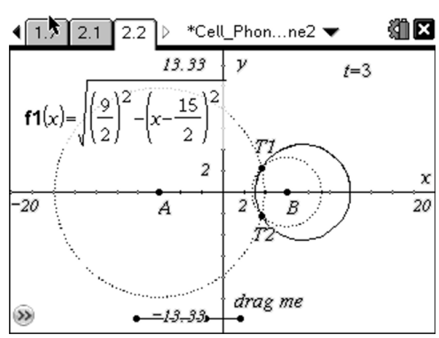
Given: TA: TB = 2:1 $\Leftrightarrow \dfrac{TA^2}{TB^2} = \dfrac{4}{1}$. Since $TA^2 = (x + a)^2 + y^2$ and $TB^2 = (x - a)^2 + y^2 \Leftrightarrow \dfrac{(x+a)^2 + y^2}{(x-a)^2 + y^2} = 4$. Simplifying this expression, we get: $\left(x - \dfrac{5}{3}a\right)^2 + y^2 = \left(\dfrac{4}{3}a\right)^2$. This is an equation of a circle with the center at point $(5/3)a$ on the *x*-axis and radius $(4/3)a$.

Students can use provided template in the TI-Nspire document to record their proofs, or they can record that on paper.

Activity extension

Problem 2 – Extension. Locus of Points T, TA:TB = t





Step 1. Students move to page 2.1, read the problem statement, make a conjecture and then move to page 2.2 to explore this situation.

Q5. Statement of the problem: Let the ratio TA: TB = t, any given number. What is the locus of point T?

Step 2. Students use the slider to observe the paths of the points T_1 and T_2. Then they can select (menu), 9: Constructions, 7: Locus. Upper semicircle will appear on the screen.

Step 3. Students repeat the step 2 for the point T_2.

Step 4. Students should vary values of *t* and observe the changes on the center of the circle and its radius. Then they should be able to formulate final conjecture.

A. . The locus is a circle with the diameter KL where KA: KB = LA: LB = *t*. If point A has coordinates $(-a, 0)$ and point B has coordinates $(a, 0)$, then the center of the circle is at $\dfrac{t^2+1}{t^2-1}a$ on the *x*-axis and the radius of the circle is $\dfrac{2ta}{t^2-1}$.

Step 5. Select (menu), View, Show Axis. The choose (ctrl) G to show function entry line and enter equation for the function, using the value of $a=6$ and given value of *t*. Observe the equation plot over the locus.

The proof is provided below:

The solution of this general case is the same as for the main problem. For example, using coordinate method, we get:

$$\frac{TA^2}{TB^2}=\frac{t^2}{1} \Leftrightarrow \frac{(x+a)^2+y^2}{(x-a)^2+y^2}=t^2 \Leftrightarrow$$

$\left(x-\dfrac{t^2+1}{t^2-1}a\right)^2+y^2=\left(\dfrac{2ta}{t^2-1}\right)^2$ - equation of the circle. It is easy to verify that when $t = 2$ we get the result above.

Activity Title Goes Here
by: author name goes here
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Materials: TI-Nspire

Cell Phone Towers
Student TI-Nspire Document: *Cell_Phone_Towers.tns*