Transformers

ID: 8772

Time required
90 minutes

Activity Overview

Students explore a special subset of the transformations of a square called the symmetry group. They also find inverses of each transformation in the symmetry group. They then delve deeper into the algebra behind transformations, connecting them with matrix multiplication. Finally, students extrapolate what they have learned about the symmetry group of a square to characterize the symmetry group of a triangle.

Topics: Transformations

- *Apply isometric transformations to polygons in the plane*
- *Use matrices to represent and apply transformations in the plane*

Teacher Preparation and Notes

- *This activity is designed for use in an Advanced Algebra 2 or Precalculus classroom.*
- *Prior to beginning the activity, students should have experience graphing polygons in the coordinate plane and some experience with simple transformations, such as vertical and horizontal shifts. Students should also be familiar with matrix multiplication and have been introduced to the concept of inverse matrices.*
- *If time constraints prevent you from completing the activity in one class period, you may choose to have students complete Problems 1 and 2 in class and Problem 3 as homework.*
- *Notes for using the TI-Nspire™ Navigator™ System are included throughout the activity. The use of the Navigator System is not necessary for completion of this activity.*
- ***To download the student TI-Nspire document (.tns file) and student worksheet, go to education.ti.com/exchange and enter "8772" in the keyword search box.***

Associated Materials

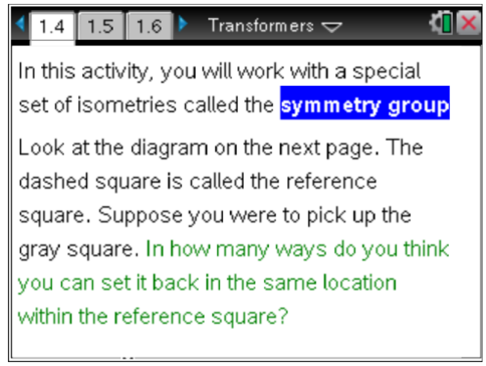
- *Transformers_Student.doc*
- *Transformers.tns*

Suggested Related Activities

To download any activity listed, go to education.ti.com/exchange and enter the number in the keyword box.

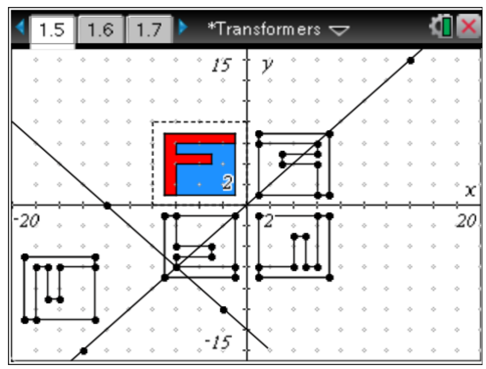
- *Transformations in Geometry (TI-84 Plus with TI-Navigator) — 1678*
- *Reflections and Rotations (TI-84 Plus family with TI-Navigator) — 8286*
- *Transformations with Lists (TI-Nspire technology) — 10278*

Problem 1 – Symmetry group for a square



In this problem, students reflect and rotate a square using the features of the *Geometry* application. The square is imprinted with the letter **F** to show students how the orientation of the square changes. (The same could be accomplished by labeling vertices.)
As students progress through pages 1.1–1.4, explain that there are an infinite number of possible transformations for any polygon. Only some of the transformations are *isometries*, or distance-preserving transformations. Discuss why some types of transformations are isometries and some are not, as well as how many different isometries are possible (an infinite number).

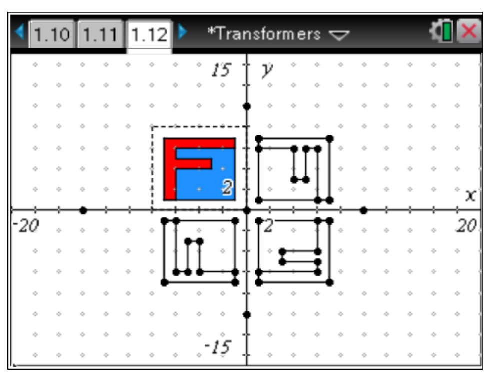
Tell students that the focus of this activity is a special subset of isometries called the *symmetry group*. Transformations in the symmetry group yield images that not only preserve distance, but also "line up" with the original polygon. A reference polygon is used to visually reinforce this definition.



After guiding students through the steps for reflecting a polygon described (page 1.7) and reviewing which reflections are allowed (page 1.8), allow them to work independently or in small groups to create reflections on page 1.5.

Students may use the **Line** tool to draw a line, reflect the figures over the line, and then slide and/or turn the line by grabbing and dragging until an allowable reflection is reached. If a reflection takes an image off of the screen, they may simply adjust the window settings in order to see it.

The challenge of this exercise is to discover that reflections over the lines $y = x$ and $y = -x$ (in addition to the x- and y-axes) are also in the symmetry group.



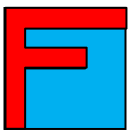
Move the discussion to rotations. After explaining the steps required to rotate a shape, described on pages 1.10 and 1.11, allow students to work as before to find the allowable rotations of the square. All of their rotations should be about the origin. Again, if an image goes off the screen, they may adjust the window settings.

The final step in Problem 1 is to find the inverse of each element of the symmetry group—that is, the transformation that "undoes" a transformation. This may be accomplished by figuring out how to transform a resulting image back onto the preimage. On page 1.5, demonstrate reflecting an image back onto the original square, and then have students work independently to find the inverses of the remaining reflections and rotations.

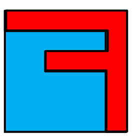
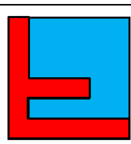
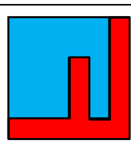
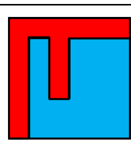
TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 1 at the end of this lesson.

Solutions

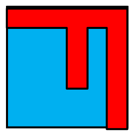
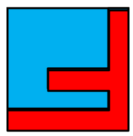
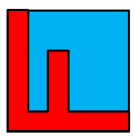
Identity

Sketch	Description	Inverse
	no change	no change

Reflections

Sketch	Description	Inverse
	reflect over $x = 0$	reflect over $x = 0$
	reflect over $y = 0$	reflect over $y = 0$
	reflect over $y = x$	reflect over $y = x$
	reflect over $y = –x$	reflect over $y = –x$

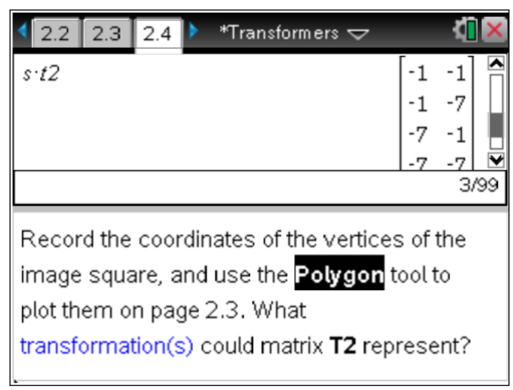
Rotations

Sketch	Description	Inverse
	rotate around origin 90° clockwise	rotate around origin 270° clockwise
	rotate around origin 180°	rotate around origin 180°
	rotate around origin 270° clockwise	rotate around origin 90° clockwise

- 8
- Sample answer: the inverse transformations are all members of the symmetry group. Also, each reflection is its own inverse.

Problem 2 – Transformer matrices

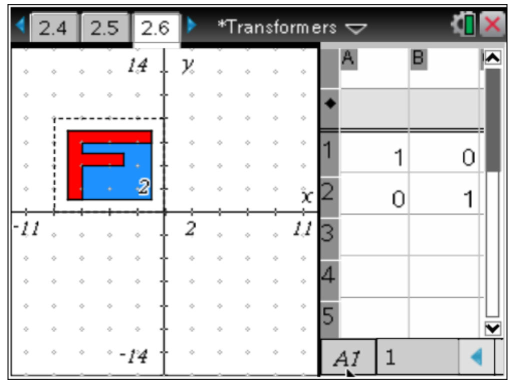
In this problem, students delve deeper into the mathematics behind these transformations. Remind them that a transformation is <u>a rule</u> for changing the coordinates of the vertices of a polygon. One way to record such a rule is in a matrix. Multiplying a matrix comprised of the preimage vertices by a "transformer matrix" yields a matrix comprised of the image vertices.



Guide students through the process of creating a matrix of vertices for the original blue square and the transformer matrix ***T2***, as given on their worksheets.

They should multiply ***S · T2*** and plot the points from the resulting matrix of vertices.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 2 at the end of this lesson.

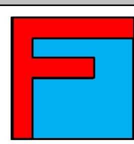
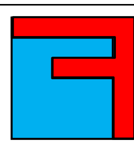
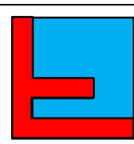
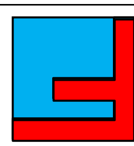


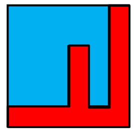
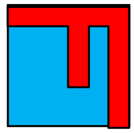
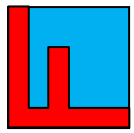
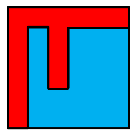
To see the effect of ***T2*** more clearly, students should enter it into the list on page 2.6. The presence of the **F** makes it clear that **T2** corresponds to reflection over the line $y = 0$. Have students work independently to match each matrix on their worksheet with the transformation it represents.

The questions on the worksheet introduce students to the idea of successive transformations. Explain that you can multiply a matrix of vertices by more than one transformer matrix to see the effect of applying more than one transformation. You can also multiply two or more transformer matrices together to create a new transformer matrix that corresponds to applying first one transformation, then the other. Students should see that multiplying the transformer matrices for transformations that are inverses yields the identity matrix.

Solutions

- $\begin{bmatrix} -7 & -7 \\ -1 & -7 \\ -1 & -1 \\ -7 & -7 \end{bmatrix}$
- reflection over $y = 0$ or rotation around origin by 270°

Transformer Matrix	Sketch	Description
$T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		no change
$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$		reflect over $x = 0$
$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$		reflect over $y = 0$
$T_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$		rotate around origin 180°

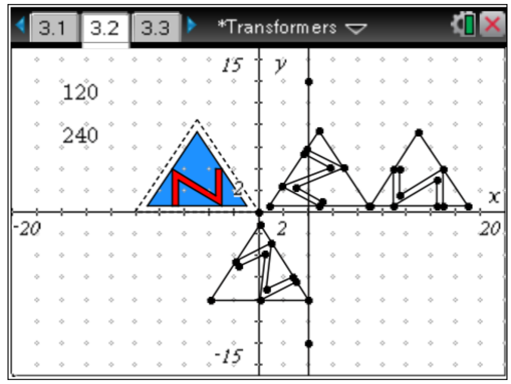
Transformer Matrix	Sketch	Description
$T_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		reflect over $y = x$
$T_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$		rotate around origin 90° clockwise
$T_6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$		rotate around origin 270° clockwise
$T_7 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$		reflect over $y = -x$

Transformer Matrix	Inverse	Transformer Matrix	Inverse
$T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$T_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$T_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
$T_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$T_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$T_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$T_6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$T_6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$T_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$T_7 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$T_7 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

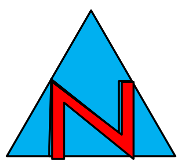
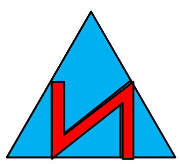
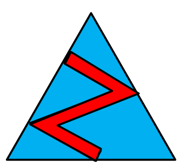
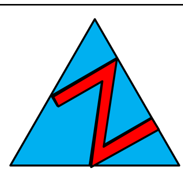
- The product of a transformer matrix and its inverse is the identity, $T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- rotating around origin 270° clockwise
- reflecting over $x = 0$

Problem 3 – Symmetry group for an equilateral triangle

In Problem 3, students use similar tools to explore a different symmetry group. As before, students should reflect and rotate the triangle, find the inverse of each transformation, find the corresponding transformer matrices, and answer the questions on the worksheet.



Solutions

Sketch	Description	Inverse	Transformer Matrix
	identity	identity	$T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
	reflect over $x = 0$	reflect over $x = 0$	$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
	rotate 120° clockwise	rotate 240° clockwise	$T_2 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$
	rotate 240° clockwise	rotate 120° clockwise	$T_3 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

TI-Nspire Navigator Opportunities

Note 1
Problem 1: *Quick Poll*
You may want to use Quick Poll to aid the discussion and ensure student understanding: Possible questions include:

- What is the inverse of the identity transformation?
- How many transformations had themselves as an inverse?
- Were any new transformations (not in the symmetry group) needed to complete the table of inverses?
- Did some transformations have more than one inverse?

Note 2
Problem 2: *Quick Poll*
You may want to use Quick Poll to aid the discussion and ensure student understanding: Possible questions include: *Is this the image of a reflection? A rotation? Can you tell? Why or why not?*