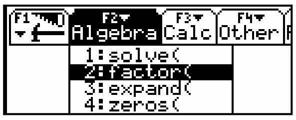
Ages 15-17 – Greatest common divisor and least common multiple
A plea for pure mathematics

Using the [F2]-Menu you can factor a number into its prime factors. Let's do this for the large number 844074000. Set your device in Exact or Auto Mode!



`factor(844074000)=` ....

(1) Using this tool you can easily find the GCD of 844074000, 4765246200 and 45585540000.
Repeat first how to find the GCD of two or more small numbers.
Hint: You can apply `factor` on a list of numbers.

844074000 = ...

4765246200 = ...

45585540000 = ..., hence GCD = ... = ...

At this moment students are using a well known algorithm for finding GCD however access to CAS enables them to deal with larger numbers and expressions than usual.

If a number is the GCD of two or more numbers, then the quotients of the given numbers and the GCD shouldn't have any prime factors in common. Check this for the previous example.

Store the GCD in your CAS: .... [STO▸] `gcd`.

Do you receive an error message? Try once more but use another name because `gcd` is obviously a reserved function name. So take:[STO▸] `cd`.

844074000 : `cd` = ... = ... (factor !)

4765246200 : `cd` = ... = ...

45585540000 : `cd` = ... = ...

Hint: A clever use of a list makes the process more comfortable!)

True? (yes / no) ....

If your answer is **no** then try to find the mistake!

(2) Can we prove the statement given in the box above?

This is a chance for the teacher to present a very easy to follow indirect proof. In earlier lessons we showed, in a short introduction into logic, that $a \to b$ *is equivalent to* $\neg b \to \neg a$.

We want to show: If $\mathrm{GCD}(a,b)=T$ then $\mathrm{GCD}\left(\frac{a}{T},\frac{b}{T}\right)=1$.

We show that $\mathrm{GCD}\left(\frac{a}{T},\frac{b}{T}\right)\neq 1$ implies that T is not the GCD of a and b.

Suppose $\mathrm{GCD}\left(\frac{a}{T},\frac{b}{T}\right)=p\neq 1$, then $\frac{a}{T}=p*\alpha$ and $\frac{b}{T}=p*\beta$.

So $a=T*p*\alpha$ and $b=T*p*\beta$ which implies that $T*p$ must be a "greater" common divisor than T. And this is obviously a contradiction to the fact that $\mathrm{GCD}(a,b)=T$.

Find and check in a similar way the GCDs of:

(3) $\mathrm{GCD}\,(328563a^3b^4c^5d^2, 247104a^2b^5c^4, 93085200a^4b^2d^6) =$ ....

(4) $\mathrm{GCD}\,(900(a+b)^3(a-b)^2(a^2-b^2)^2, 1000(a^2+b^2)(a^2-b^2)^2(a^3-b^3),$
$500(a^3-b^3)^2(a^3+b^3)^2(a^2-b^2)^2) =$ ....

Comment: It seems that it is not possible to factor general expressions and the coefficients in one operation. Try to find an alternative!

(5) $\mathrm{GCD}\,(36a^6+12a^5b-47a^4b^2-14a^3b^3+12a^2b^4+2ab^5-b^6,$
$54a^8-27a^7b-63a^6b^2+34a^5b^3+8a^4b^4-7a^3b^5+a^2b^6,$
$18a^5b^2-39a^4b^3+20a^3b^4+6a^2b^5-6ab^6+b^7)$

GCD = ...

These examples will force the students to check their data. They have to edit the bulky expressions very carefully and copy the results onto paper.

The device "knows" the function `gcd(number1,number2)` for calculating the GCD of *number*1 and *number*2. Check this using two randomly chosen pairs of numbers.

(6) Calculate GCD(844074000, 45585540000, 4765246200) and compare with (1)!

It was interesting to observe students trying to find a way out of that trap. Eventually they discovered the concept of a nested function and immediately tried to extend this for four or more numbers.

How could you use `gcd` for three or more numbers? ……..

(7) What is the meaning of the abbreviation `gcd`?

....

Have you tried `gcd` for non-integer numbers? Do that now and report your results:

GCD(.....................................) = ...

GCD(.....................................) = ...

GCD(.....................................) = ...

Here we encourage the students to generalize a well known rule in an unusual way. They are asked to interpret the outcomes.

Explain the outcome. Can you find a rule?

I assume that students used only rational numbers. Now consider at least one irrational number:

GCD(......................................) = ...

(8) I also assume that students have worked with decimal numbers. It is "more mathematical" to work with fractions:

$$\mathrm{GCD}\left(\frac{3}{4},\frac{1}{2}\right) = \text{.............................}$$

$$\mathrm{GCD}\left(\frac{3}{4},\frac{1}{2},\frac{3}{5}\right) = \text{.............................}$$

$$\mathrm{GCD}\left(\frac{3}{4},\frac{1}{2},\frac{3}{5},\frac{17}{3}\right) = \text{.............................}$$

The conjecture is that the result is a fraction $\frac{1}{\mathrm{lcm}}$ (all denominators).

Does your conjecture also hold for $\mathrm{GCD}\left(\frac{3}{4},\frac{6}{5}\right)$?

Now we are learning that our first idea might not hold. We have to refine the interpretation..

Is a new interpretation necessary?

Can you find a problem which needs $\mathrm{GCD}\left(\frac{3}{4},\frac{6}{5}\right)$ for its solution?

The rule is gcd(numerators)/lcm(denominators). It is the greatest fraction which gives integer results for the divisor for the given numbers.

Problem: Given is a part of a wall 0.75 × 1.20 which should be covered by tiles in form of a square of maximum area. How many tiles of which size are necessary?

Solution: $\mathrm{GCD}\left(\frac{3}{4},\frac{6}{5}\right) = \frac{3}{20}$; $\left(\frac{\frac{3}{4}\cdot\frac{6}{6}}{\left(\frac{3}{20}\right)^2}\right) = 40$.

We need 40 tiles with side length 3/20 = 0.15.

We can assume that the students know how to find the LCM of two (smaller) numbers.

(9) The next task is to find the least common multiple for the numbers from (1) by-hand: (using `factor()` is allowed)

LCM (844074000, 4765246200, 45585540000) = ...

The main task in the next paragraph is to find a check for LCM and to explain the process and check the validity of any assumptions made.

(10) Consider a check similar to that for the GCD. (Refer to task (2)).

Describe the process:

Confirm your conjecture using the numbers from (9). Try to find the proof by yourself.

Find the LCMs for the expressions from (3), (4) and (5). Check the result for at least one of them:

(11) LCM(problem 3) = ...

(12) LCM(problem 4) = ...

(13) LCM(problem 5) = ...

Task (14) is not so demanding for English speaking students. But in Austria we call the LCM "kleinstes gemeinsames Vielfaches" with the abbreviation kgV. So this task forces them to use the manual and look carefully for the required CAS function.

(14) Which function of your calculator returns the LCM of two numbers? Try it and apply again the nested function concept. Check one of the results of (9), (11), (12) or (13)
....
(Use the manual!)

(15) Try to solve the following problem.
Wine is bottled in bottles to 0.5 litres. It shall be poured into other bottles of 0.35 litres content. How many bottles à 0.35 litres must be used that no rest will remain in the 0.5 l bottles?
(*provided by René Hugelshofer)*

(16) What is the meaning of LCM for rational numbers?

(17) Try again (15).

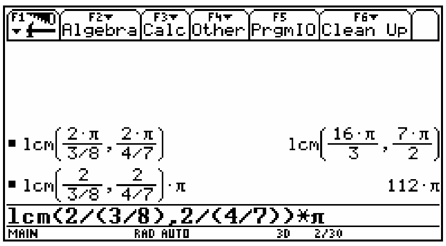
<u>Solution</u>*:* $\mathrm{LCM}\left(\frac{1}{2},\frac{35}{100}\right)=\frac{7}{2};\quad \frac{\frac{7}{2}}{\frac{35}{100}}=10$ and $\frac{\frac{7}{2}}{\frac{1}{2}}=7$

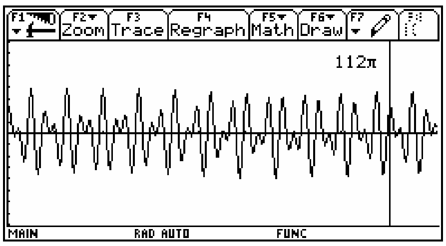
10 bottles of 0.35 litres are filled by 7 bottles of 0.5 litres.

The LCM of two fractions has another fine application which unfortunately cannot be explained at this level. (Idea from Guido Herweyers).

What is the period of $\sin\left(\frac{3x}{8}\right)+\cos\left(\frac{4x}{7}\right)$?

<u>Solution</u>*:*





(18) The LCM of two expressions *a*,*b* can easily be derived from the GCD(*a*,*b*) and the two expressions. Find this relationship!

Work systematically! Start your investigation using smaller numbers and try to find a rule.

a = ...; *b* = ..., GCD(*a*,*b*) = ..., LCM(*a*,*b*) = ...

a = ...; *b* = ..., GCD(*a*,*b*) = ..., LCM(*a*,*b*) = ...

a = ...; *b* = ..., GCD(*a*,*b*) = ..., LCM(*a*,*b*) = ...

a = ...; *b* = ..., GCD(*a*,*b*) = ..., LCM(*a*,*b*) = ...

There are so many patterns in mathematics and from my point of view recognizing patterns can very often help students solve their maths problems, so we need to give students many more opportunities to practice this ability to recognise patterns.

The next step is to express the pattern in words, then as a rule or formula.

When you are sure you know the relationship you should check your conjecture using two "larger" numbers or expressions from the 1st sheet:

You should be able to express the relationship as a formula:

GCD(*a*,*b*) =

Express this formula in words:

Until now the conjecture has been an unproved conjecture and not a theorem. The proof is still lacking. Try to carefully follow the exact proof shown (*Eugenio Roanes gave a sketch of this proof after a talk in Gettysburg.*).

Take two numbers *a* = 8400 and *b* = 1440

$a = 2^4 \cdot 3 \cdot 5^2 \cdot 7$ and $b = 2^5 \cdot 3^2 \cdot 5 \cdot (7^0)$

GCD(8400, 1440) = $2^4 \cdot 3^1 \cdot 5^1 \cdot 7^0$ and LCM(8400, 1440) = $2^5 \cdot 3^2 \cdot 5^2 \cdot 7^1$

GCD $*$ LCM = $2^{4+5} \cdot 3^{1+2} \cdot 5^{1+2} \cdot 7^{0+1} = a * b$

Now we will generalize the process shown above.

$a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot$ .... $\cdot p_n^{\alpha_n}$ and $b = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot$ .... $\cdot p_n^{\beta_n}$

with p_n = greatest prime factor occuring. If p_k does not occur in one factorization then α_k or $\beta_k = 0$).

GCD(*a*,*b*) = $p_1^{\min(\alpha_1,\beta_1)} \cdot p_2^{\min(\alpha_2,\beta_2)} \cdot$ .... $\cdot p_n^{\min(\alpha_n,\beta_n)}$

LCM(*a*,*b*) = $p_1^{\max(\alpha_1,\beta_1)} \cdot p_2^{\max(\alpha_2,\beta_2)} \cdot$ .... $\cdot p_n^{\max(\alpha_n,\beta_n)}$ hence

GCD(*a*,*b*) *LCM(*a*,*b*) $= p_1^{\min(\alpha_1,\beta_1)+\max(\alpha_1,\beta_1)} \cdot p_2^{\min(\alpha_2,\beta_2)+\max(\alpha_2,\beta_2)} \cdot$ .... $\cdot p_n^{\min(\alpha_n,\beta_n)+\max(\alpha_n,\beta_n)}$

$= p_1^{\alpha_1+\beta_1} \cdot p_2^{\alpha_2+\beta_2} \cdot$ .... $\cdot p_n^{\alpha_n+\beta_n}$

This was a welcome opportunity to present an exact proof using mathematical notation. It was helpful to refer to the example above at each stage of the proof so that students could follow the process.

Then I had the- crazy - idea to find another generalization: is it possible to extend the theorem for more than two numbers?

The students and I encountered a real challenge in exploring and investigating the next problem. We were very surprised to find a solution using trial and error.

(19) The above theorem might imply the consequence that we could generalize for three numbers/expressions: $\boldsymbol{a * b * c = \mathrm{LCM}(a,b,c) * \mathrm{GCD}(a,b,c)}$ **??**

Check if this statement is true or not for two examples of your choice.

(20) Try the numbers 12, 30 and 64.

Are there any triples of numbers for which the formula holds???

(21) Try for (12, 18, 25).

(22) See if the formula holds for (150,210,330).

Explain your results.

Sometimes you can have a "mathematical feeling" but it is not so easy to express that feeling in words. It is very important to practice the ability to explain mathematics.

(23) Here is a real challenge.

Try to find a relation between GCD and LCM which holds in all cases:

The students found one solution:

$$a\,b\,c = \frac{\mathrm{LCM}\,(a,b,c)\cdot \mathrm{GCD}(a,b)\cdot \mathrm{GCD}(a,c)\cdot \mathrm{GCD}(b,c)}{\mathrm{GCD}(a,b,c)}.$$

At the time when I was busy with this paper in school I was running a T^3 *seminar for Secondary School teachers and I presented this paper as an example for working with the TI-92. At 10 30 in the evening my phone rang and a colleague proudly told me that she had found a solution:*

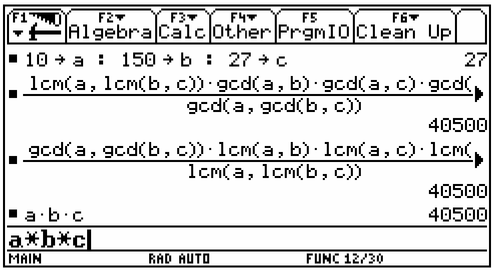
$$a\,b\,c = \mathrm{LCM}(a,b,c)\cdot \mathrm{GCD}(a,b,c)\cdot \mathrm{LCM}(\mathrm{GCD}(a,b),\mathrm{GCD}(a,c),\mathrm{GCD}(b,c)).$$

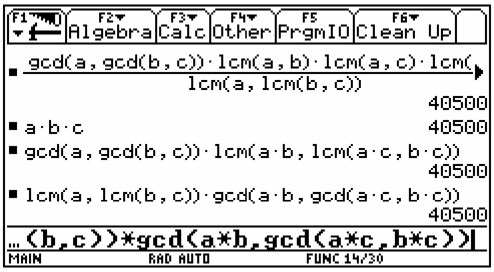
I wrote an e-mail to my friend Hannes Wiesenbauer and asked him for an alternative solution. He replied: "That problem is not so easy to solve, but there is a solution:" and he added a theorem which leads to

$$abc = \mathrm{GCD}(a,b,c)\cdot \mathrm{LCM}(ab, ac, bc)$$
$$abc = \mathrm{LCM}(a,b,c)\cdot \mathrm{GCD}(ab, ac, bc)$$

The two formulae show a wonderful symmetry and we could observe the principle of duality which the students had met a couple of weeks before when they had a short introduction into set theory and logic.

You might imagine that it did not last very long and they asked if there was a dual formula for their product and fortunately there was.

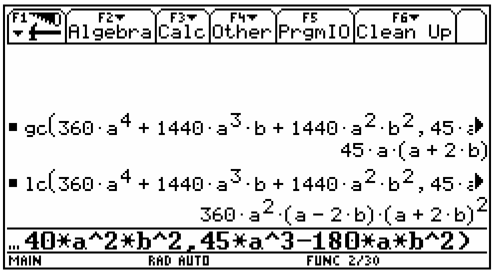




(24) Unfortunately there is no function in the TI-92 which allows students to find the GCD and LCM of general expressions. So now we will try to develop such a function.

Consider two functions `gc` und `lc`, which are working in a way presented on the TI-screen shown below. You should find the GCD and LCM of the expressions

$360a^4 + 1440a^3b + 1440a^2b^2$ and $45a^3 - 180ab^2$



As `gc` and `lc` are related very closely (see (18)):

....,

It will be sufficient to find a function for one of them.

<u>Hint</u>: Do you know any application of GCD and/or LCM?

We will stick to the LCM.

The TI-function `getdenom(fraction)` gives back the denominator of a fraction.
Try this for the following fractions:

What are the denominators of the fractions: $\frac{2}{5}, \frac{2a}{5b}, \frac{a^2+b}{b^2+a}, 5a, \frac{3}{4}\cdot\frac{2}{9}$

Use [CATALOG] to find the respective function which returns the numerator of a fraction:

....

Check it: ....

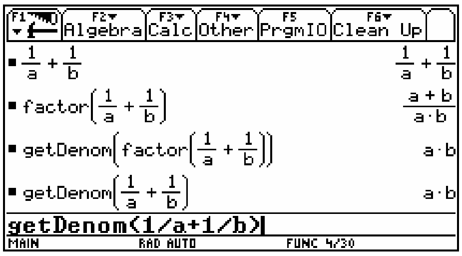
`getdenom` will support our task to find the function `lc`!

Are you ready?

Do you have a formula, which returns the LCM of two expressions `t1` and `t2`?

Then define your function in the following way:

`define lc(t1,t2)` = ...



You can follow the learning process to extract the denominator of a sum of fractions, which seems to form the lcm of the two expressions t1 and t2. The students learned to develop and finally to define a self made function (and later to use it). It is very interesting to observe the strategies of the students See a possible way to achieve the solution.

Check the validity of your function by applying it to the two expressions from the screenshot.

Does your function work?

Congratulations! But to make sure, perform just one more little test.

Check your function for two numbers: `lc(3,6)` = ...

Is everything ok? Why not?? What has happened??

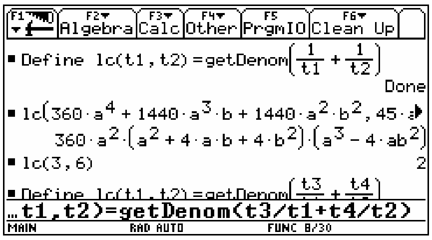
Find an explanation and improve the function.

Students should learn not be satisfied with their results too early. It makes sense to check the validity of rules even for very trivial cases.

Improved version (*Introducing auxiliary variables* `t3` *and* `t4` *to avoid cancellation*):

`define lc(t1,t2)` =

....
....



We can use one function to define a new one, which gives:

`define gc(t1,t2)` = ...

Closing comments

I like applied mathematics but I am sure that we can not do without pure mathematics. I found that the students liked investigating the problems presented and I spent a lot of time during the breaks answering their questions (What if, Why not? Am I right? Do you think?....). I didn't hear the question: What is all that stuff good for? I was teacher at a vocational school!!! So I think we can tempt pupils to have their "Adventures in Their Heads". It is important to explain what they have done, for example: "Now you found an important rule on your own!", "This is your first self made function!",... The students are really proud of their mathematical achievements and their selfconfidence is growing. There were some very emotional moments when I felt that the sparkles were spreading and when I then left the classroom I felt something like: "That is it, I´ve got them". These are the precious moments when we learn why we have become teachers. I know that the use of a CAS in whatever form is not a necessary condition, but it is helpful in providing a new motivation for students and, for teachers too. We should not let these unique chances for learning pass us by. *What I have shown here using the TI-92 is platform independent. I am sure that many chapters of our traditional curricula can be changed and that it is not necessary to add many new topics as a* *consequence of the use of modern technologies. On the contrary, I must say, that it is likely that we now need more time in the classroom than before, because mathematics teaching has a much wider scope than ever before. New technologies not only open new fields for school mathematics and enable us to tackle interesting applications but may also sometimes help us to find a way "back to the roots".*

Two Assessment Examples

The students used their self defined function kgv to check the common denominator for calculations with fractions and later for solving equations. Students also did this using traditional methods simple problems.

(1) This example shows how to combine indispensable basic skills and computer algebra.
(C4 at this level, because factoring the denominators is too time consuming.)

$$\frac{6}{x^3+4x^2-9x-36} - \frac{2}{2x^3+19x^2+59x+60} = \frac{9}{2x^3+5x^2-18x-45}$$

a) Use the TI to factor the denominators.
b) Find the domain.
c) Solve the equation by-hand.

The second example demonstrates that using a computer algebra system addresses much more than manipulation skills.

(2) Symbolic Algebra in Equations (C3)

$$\frac{b-x}{a^2} + \frac{a+x}{b^2} = \frac{b}{a^2} - \frac{3x-a}{b^2} + \frac{b+2a}{a^2b^2}$$

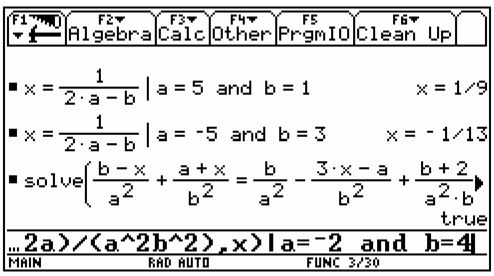
a) Enter the equation and explain the TI´s "simplification".
b) Give the equation´s general solution and state the necessary conditions for the solution to be valid.
c) Find two special solutions!
d) Write down the output from the TI.
e) Explain the second part of the output and give an example to show why this output was obtained.

Answers (for question 2):

a) Do the first part by-hand (bring over a common denominator)

b) $x = \dfrac{1}{2a-b}$ with $a,b \neq 0$ and $b \neq 2a$

c) $(a = 5, b = 1) \rightarrow \; x = 1/9$
$(a = -5, b = 3) \rightarrow \; x = -1/13$



d) Copy the expression

e) For $b=-2a$ the equation turns out to be an identity.

f) This needs manual calculation:

Multiplication of the equation by a^2b^2 and later $x(4a^2-b^2)=2a+b\,|:(2a+b)\neq 0$