EXPLORATIONS

Activity 6

The Tortoise and the Hare

Teacher Notes

Concepts

- Addition and subtraction of fractions
- Multiplication of fractions
- Decimals
- Algebraic representation
- Infinite processes with limits

Calculator Skills

- Fraction operations: [A b/c], [2nd] [F◂▸D], [2nd] [A b/c◂▸d/e]
- Using variables: [STO▸], [MEMVAR]
- Using Constant mode: [2nd] [K]

Materials

- TI-30X IIS
- Student Activities pages (p. 60-62)
- Scissors
- 2 sheets of 8 ½″ by 11″ paper

Objective

- In this activity, students will develop patterns using two or more rational number quantities. By understanding the relationship between these quantities and the sums of these quantities, they will understand functions in more formal courses of study in high school.

Topics Covered

- Applying mathematical models to problem situations
- Understanding the conceptual foundation of limit
- Analyzing patterns in rational expressions and in sums of rational expressions

Introduction

An age-old story exists about a tortoise and a hare. The two begin to race and the faster hare stops halfway through the race to take a nap. The tortoise, not to be outdone, continues slowly along until the hare is awakened by the tortoise’s steps. The hare decides that having such a decided advantage, he will run half the distance remaining from the spot of his nap to the finish line and then take another rest. The hare continues his plan for the remainder of the race.

Who will win the race?

What fractional part of the distance from START to FINISH had the hare run when he was on his sixth nap?

How much of the distance remained for the hare when he awakened from his sixth nap?

We can model this problem in a different way to help us arrive at answers to these questions.

Investigation

1. Take a sheet of 8½" by 11" paper, and fold it exactly in half. Use scissors to cut along the crease so that you now have two equal halves of paper. What fractional part of the whole sheet does each piece represent?

   Demonstrate the result on the overhead calculator. Since you will be folding and cutting in one-half each time, begin by defining "multiply by 1/2" as a constant operation.

Press:	The calculator shows:
[CLEAR] [2nd] [K] [×] 1 [A b/c] 2 [ENTER]	K =*1⌟2 DEG K
1 [ENTER]	1 * 1⌟2 1/2 DEG K

2. Lay one of the halves aside on your desk and fold the remaining piece exactly in half. Cut along the crease as before. What fractional part of the original whole sheet of paper does each of these pieces represent? Demonstrate this step on the calculator.

Press:	The calculator shows:
[ENTER]	Ans * 1⌟2 ¼ DEG K

3. Lay one of these pieces (one-fourth) next to the one-half piece on your desk as if you were reconstructing the whole sheet all over again. Take the remaining one-fourth sheet, and fold it exactly in half and cut along the crease. Continue this process of folding, cutting, and reconstructing until the pieces of paper become too small. Also, find each result on the calculator as you proceed and complete the Student Activity page.

Press:	The calculator shows:
[ENTER]	Ans * 1⌟2 1/8 DEG K
[ENTER]	Ans * 1⌟2 1/16 DEG K

Note that the calculator will display unit fractions (fractions with a numerator of '1' in common fraction form) until it gets to $^1/_{1024}$. The TI-30X IIS displays this number as 0.000976563. Each successive multiplication by ½ from then on will be displayed in decimal form.

4. Use the calculator to represent the SUM of the pieces in the reconstructed rectangle. Clear the value stored in variable A and the previously defined constant calculation. Then define a new constant:

   *$^1/_2$ + A, where A is $^1/_2$

Press:	The calculator shows:
[2nd] [K] [2nd] [K] [CLEAR]	K = DEG K
[×] 1 [A b/c] 2 [+] [MEMVAR] [ENTER] [ENTER]	K = 1 * 1⌟2 + A DEG K
1 [ENTER]	K = 1 * 1⌟2 + A 1/2 DEG K
[STO▸] [ENTER]	Ans → A 1/2 DEG K
[ENTER]	Ans * 1⌟2 + A 3/4 DEG K

Continue to press [ENTER] very slowly, complete Student Activity page, and observe the sequence of fractions that are displayed.

5. Use the results on the Student Activity page to help answer the questions about the tortoise and the hare. What sum does the "SUM OF PIECES" entry seem to be approaching as the denominators get larger and as more fractions are added?

6. Investigate the fractional powers of 3 and 4 with your students. Use your calculator to complete Student Activity Part 2 and Student Activity Part 3 and observe the patterns for $\frac{1}{3}$ and for $\frac{1}{4}$. Using decimal approximations on the calculator may help to see what number the sum is approaching. (Note: Be sure to clear variable A and the constant operation before completing Student Activity Part 2 and Student Activity Part 3.)

Wrap-Up

- The pattern of powers for the $\frac{1}{3}$ and $\frac{1}{4}$ fractions should be easily predicted by the class. Graphing the pattern of $\frac{1}{n}$ can help lead to a discussion of why division by 0 is not defined. The graph can be a very powerful tool for illustrating this concept.
- Remind the students to press [2nd] [K] to turn off Constant mode when they complete this activity.

Extensions

A unit fraction is a fraction of the form $\frac{1}{n}$, where *n* is a positive whole number. Investigate whether a unit fraction can be written as a sum of two different unit fractions. Use $\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$ $\frac{1}{3}=\frac{1}{4}+\frac{1}{12}$ $\frac{1}{4}=\frac{1}{5}+\frac{1}{20}$

1. Describe the pattern and write the algebraic representation of your pattern.

   $\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n \times (n+1)}$

   Test your pattern to see if it works on the three examples above.

2. Find a way to write 1/8 as the sum of two different unit fractions.

   $\frac{1}{12}+\frac{1}{24}$ *or* $\frac{1}{10}+\frac{1}{40}$ *(answers will vary)*

3. Can you find five different unit fractions whose sum is 1?
   Hint: For a sum of four unit fractions whose sum is 1, try this:

   $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}=1$

   $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{12}+\frac{1}{24}$

   $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{10}+\frac{1}{40}$

   (answers will vary)

Solutions Part 1

Step	Area of pieces in the reconstruction	Sum of pieces	Area of piece left at each step
Start	0	= 0	0
1.	$\frac{1}{2}$	$= \frac{1}{2}$	$\frac{1}{2}$
2.	$\frac{1}{2} + \frac{1}{4}$	$= \frac{3}{4}$	$\frac{1}{4}$
3.	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$= \frac{7}{8}$	$\frac{1}{8}$
4.	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$	$= \frac{15}{16}$	$\frac{1}{16}$
5.	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$	$= \frac{31}{32}$	$\frac{1}{32}$
6.	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$	$= \frac{63}{64}$	$\frac{1}{64}$
7.	$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{64} + \frac{1}{128}$	$= \frac{127}{128}$	$\frac{1}{128}$
8.	$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{64} + \frac{1}{128} + \frac{1}{256}$	$= \frac{255}{256}$	$\frac{1}{256}$
9.	$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{128} + \frac{1}{256} + \frac{1}{512}$	$= \frac{511}{512}$	$\frac{1}{512}$
10.	$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}$	$= \frac{1023}{1024}$	$\frac{1}{1024}$
n	$\frac{1}{2^1} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$	$= \frac{2^n - 1}{2^n}$	$\frac{1}{2^n}$

Solutions Part 2

Step	Pattern of powers of one-third	Sum at this step	Decimal approximation
Start	0	= 0	0
1.	$\frac{1}{3}$	$= \frac{1}{3}$	0.333...
2.	$\frac{1}{3} + \frac{1}{9}$	$= \frac{4}{9}$	0.444...
3.	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27}$	$= \frac{13}{27}$	0.481481...
4.	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$	$= \frac{40}{81}$	0.49382716...
5.	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$	$= \frac{121}{243}$	0.497942387...
6.	$\frac{1}{3} + \frac{1}{9} + \ldots + \frac{1}{243} + \frac{1}{729}$	$= \frac{364}{729}$	0.499314129...
7.	$\frac{1}{3} + \frac{1}{9} + \ldots + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187}$	$= \frac{1093}{2187}$	0.499771376...
8.	$\frac{1}{3} + \frac{1}{9} + \ldots + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561}$	$= \frac{3280}{6561}$	0.499923792...
n	$\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n}$	$= \frac{\left(\frac{3^n - 1}{2}\right)}{3n}$	$\approx$0.5 (approximately .5)

Solutions Part 3

Step	Pattern of powers of one-fourth	Sum at this step	Decimal approximation
Start	0	= 0	0
1.	$\frac{1}{4}$	$= \frac{1}{4}$	0.25
2.	$\frac{1}{4} + \frac{1}{16}$	$= \frac{5}{16}$	0.3125
3.	$\frac{1}{4} + \frac{1}{16} + \frac{1}{64}$	$= \frac{21}{64}$	0.328125
4.	$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}$	$= \frac{85}{256}$	0.3320313
5.	$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024}$	$= \frac{341}{1024}$	0.333007813
6.	$\frac{1}{4} + \frac{1}{16} + \ldots + \frac{1}{1024} + \frac{1}{4096}$	$= \frac{1365}{4096}$	0.333251953
7.	$\frac{1}{4} + \frac{1}{16} + \ldots + \frac{1}{4096} + \frac{1}{16384}$	$= \frac{5461}{16384}$	0.333312988
8.	$\frac{1}{4} + \frac{1}{16} + \ldots + \frac{1}{16384} + \frac{1}{65536}$	$= \frac{21875}{65536}$	0.333328247
n	$\frac{1}{4^1} + \frac{1}{4^2} + \ldots + \frac{1}{4^n}$	$= \frac{\left(\frac{4^n - 1}{3}\right)}{4^n}$	Approximately 0.33333333

Student Activity 6

Name ______________________________

Date ______________________________

Rational Number Patterns — The Tortoise and the Hare

Objective: *In this activity, you will develop patterns using two or more rational number quantities. By understanding the relationship between these quantities and the sums of these quantities, you will be able to understand functions in more formal courses of study in high school.*

Part 1: Finding Patterns with Powers of $\frac{1}{2}$

Use the paper model created in the investigation and your calculator to generate the entries in the table. The first few entries have been made for you.

Step	Area of pieces in the reconstruction	Sum of pieces	Area of piece left at each step
Start	0	= 0	0
1.	$\frac{1}{2}$	$= \frac{1}{2}$	$\frac{1}{2}$
2.	$\frac{1}{2} + \frac{1}{4}$	$= \frac{3}{4}$	$\frac{1}{4}$
3.	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$= \frac{7}{8}$	$\frac{1}{8}$
4.	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$	=	$\frac{1}{16}$
5.	$\frac{1}{2} +$	=	
6.		=	
7.		=	
8.		=	
9.		=	
10.		=	
n		=	

Part 2: Finding Patterns with Powers of $\frac{1}{3}$

Step	Pattern of powers of one-third	Sum at this step	Decimal approximation
Start	0	= 0	0
1.	$\frac{1}{3}$	$= \frac{1}{3}$	0.333...
2.	$\frac{1}{3} + \frac{1}{9}$	$= \frac{4}{9}$	0.444...
3.	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27}$	$= \frac{13}{27}$	0.481481...
4.	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$	=	
5.		=	
6.		=	
7.		=	
8.		=	
n		=	

Part 3: Finding Patterns with Powers of $\frac{1}{4}$

Step	Pattern of powers of one-fourth	Sum at this step	Decimal approximation
Start	0	= 0	0
1.	$\frac{1}{4}$	$= \frac{1}{4}$	0.25
2.	$\frac{1}{4}+\frac{1}{16}$	$= \frac{5}{16}$	0.3125
3.	$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$	$= \frac{21}{64}$	0.328125
4.	$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$	$= \frac{85}{256}$	0.3320313
5.		=	
6.		=	
7.		=	
8.		=	
n		=	