

## Chapter 6

### Applications of Integrals

In this chapter, you will use the TI-89 to investigate various applications of integration. In some cases, you will use a symbolic approach; but in other cases where there may be no closed analytic solution, you will use graphical and numerical methods.

#### Example 1: Area between two curves

Find the area bounded by  $y = e^{x^2} - 2$  and  $y = \cos(x)$ .

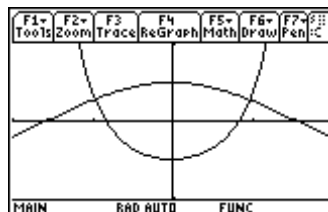
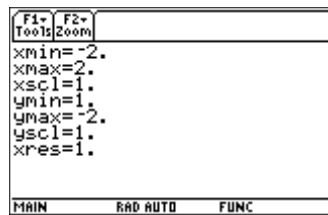
#### Solution

First, graph the two equations. The area is given by

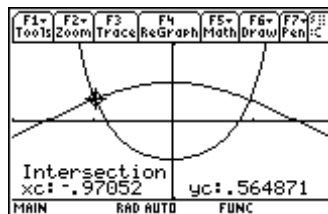
$$\int_a^b y_2(x) - y_1(x) dx$$

Then use the graph to get the values for the left and right intersection points. With these values, you can evaluate the definite integral.

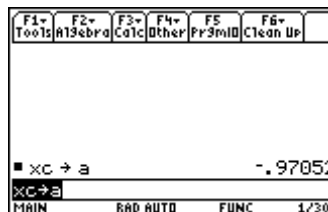
1. Press  $\boxed{2nd}$   $\boxed{[F6]}$  **Clean Up** and select **2:NewProb** to clear variables and set other defaults.
2. Set **Exact/Approx=AUTO** on Page 2 of the MODE dialog box.
3. In the Y= Editor, enter the equations in  $y_1$  and  $y_2$ .
4. Set the Window variable values as shown for a  $[-2,2] \times [-2,2]$  viewing window.
5. Graph the equations.



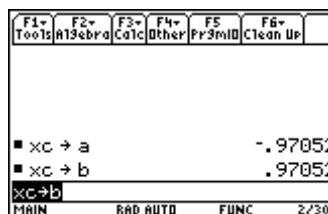
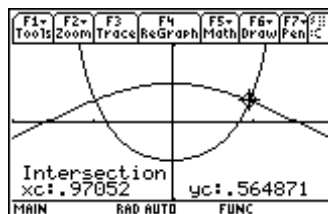
6. Find the left point of intersection. Press **[F5] Math** and select **5:Intersection**. Press **[ENTER]** to use  $y_1$  as the first curve, and press **[ENTER]** again to use  $y_2$  as the second curve. Press **[◀]** and **[▶]**, or type values, to set the bounds.



7. Press **[HOME]** to return to the Home screen and store the  $x$ -coordinate as  $a$  for later use in the definite integral.

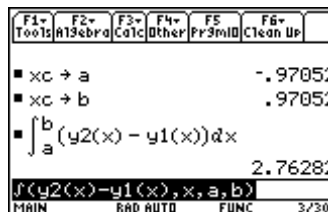


8. Return to the graph screen and repeat steps 6 and 7 to find and store the right point of intersection.



9. Evaluate the definite integral.

**[2nd] [∫] Y2 ( X ) - Y1 ( X ) , X , A , B ) [ENTER]**



The area between the two curves is about 2.76282.

### Example 2: Arc length

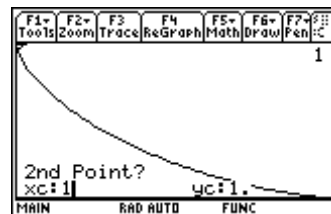
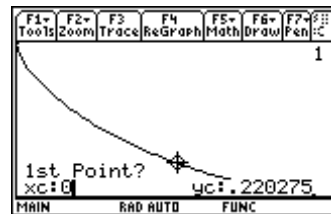
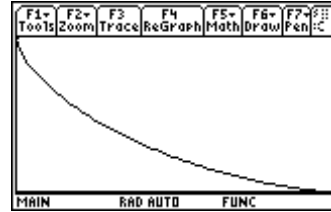
Find the length of the astroid  $y = (1 - x^3)^{2/3}$  in the first quadrant.

#### Solution

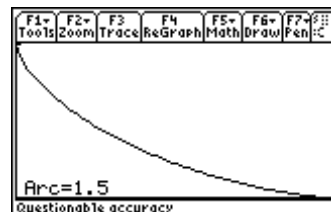
First, find the length of a curve using the built-in Arc feature of the TI-89. Then compare the result with the definite integral for arc length, where arc length is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

1. Press **2nd** **[F6]** **Clean Up** and select **2:NewProb** to clear variables and set other defaults.
2. Enter the equation in *y1* in the Y= Editor and clear *y2*. Set the Window variable values for a [0,1] x [0,1] viewing window. Then graph *y1*.
3. Press **[F5]** **Math** and select **8:Arc**. You are prompted for the *x*-coordinates of the left and right endpoints of the arc.
4. Enter **0** for the *x*-coordinate of the first point (*xc*) and **1** for the *x*-coordinate of the second point. You can ignore the *y*-coordinate values (*yc*).

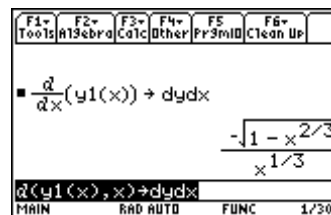


The length of the arc is 1.5. Now compare this result with the definite integral result.



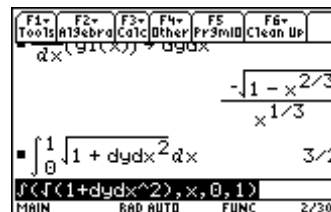
5. Store the derivative of *y1* in the variable *dydx* to make it easier to enter the definite integral.

**2nd** **[d]** **Y1** **(** **X** **)** **,** **X** **)** **STO>** **DYDX** **ENTER**



6. Enter the expression for the definite integral.

**2nd** **[∫]** **2nd** **[√]** **1** **÷** **DYDX** **^** **2** **,** **X** **,** **0** **,** **1** **)** **ENTER**



This result confirms the value we found using the **Arc** feature of the **Math** menu.

### Example 3: First-order differential equations

The next few simple examples illustrate applications of integration that can be solved using the Differential Equation Solver (**deSolve**) of the TI-89.

Solve the differential equation

$$y' = 9.8t + 2.7$$

#### Solution

1. Press  $\boxed{2\text{nd}} \boxed{F6}$  **Clean Up** and select **2:NewProb** to clear variables and set other defaults.
2. Find the solution using the **deSolve**( command.

$\boxed{\text{CATALOG}} \boxed{\text{deSolve}} \boxed{(Y} \boxed{2\text{nd}} \boxed{[']} \boxed{=}$   $9.8$   $T$   $+$   $2.7$   $,$   $T$   $,$   $Y$   $\boxed{)}$   
 $\boxed{\text{ENTER}}$

The solution is

$$y = 4.9t^2 + 2.7t + C$$

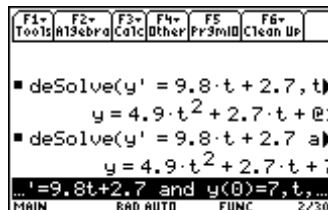
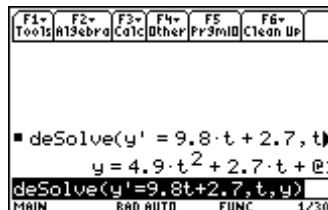
Notice that the constant of integration is represented with the  $\text{C}$  1 symbol.

3. To solve the same differential equation with initial conditions  $y(0) = 7$ :

Edit the **deSolve**( command as follows:

**deSolve**( $y'=9.8t+2.7$  and  $y(0)=7,t,y$ )

**Note:** The **and** operator is in the CATALOG.



### Example 4: Second-order differential equations

A ball was tossed straight up from an initial height of 0.29 meters and with an initial velocity of 3.8 m/s. Solve the second-order differential equation

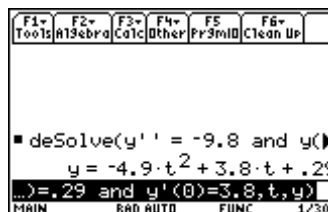
$$y'' = 9.8, \quad y(0) = .29, \quad y'(0) = 3.8$$

to find an equation to model the height of the ball over time.

#### Solution

Clear the calculator and enter the **deSolve**( command.

$\boxed{\text{CATALOG}} \boxed{\text{deSolve}} \boxed{(Y} \boxed{2\text{nd}} \boxed{[']} \boxed{2\text{nd}} \boxed{[']} \boxed{=}$   $(-)$   $9.8$   $\boxed{\text{CATALOG}} \boxed{\text{and}}$   
 $\boxed{Y} \boxed{(}$   $0$   $\boxed{)}$   $\boxed{=}$   $0.29$   $\boxed{\text{CATALOG}} \boxed{\text{and}}$   $\boxed{Y} \boxed{2\text{nd}} \boxed{[']} \boxed{(}$   $0$   $\boxed{)}$   $\boxed{=}$   $3.8$   $,$   
 $T$   $,$   $Y$   $\boxed{)}$



**Example 5: Scatter plots and regression curves**

We measured the height of the ball in Example 4 with a Texas Instruments CBL™. Here is the data from the CBL.

Time (seconds)	Height (meters)
0	.29
.04	.44
.08	.57
.12	.68
.16	.78
.20	.86
.24	.92
.28	.96
.32	.99
.36	1.0

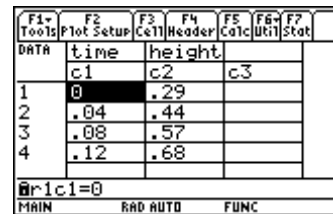
Time (seconds)	Height (meters)
.40	1.0
.44	1.0
.48	.97
.52	.94
.56	.89
.60	.82
.64	.74
.68	.64
.72	.53
.76	.40

Continued in next column

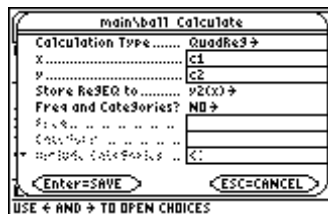
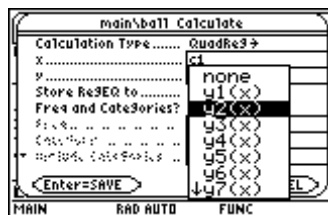
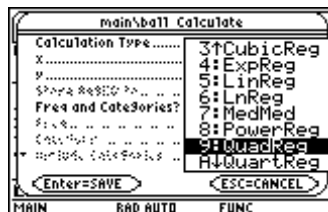
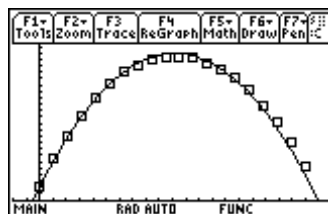
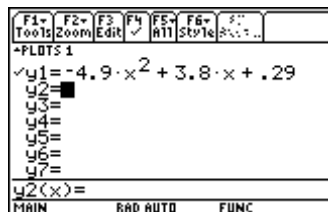
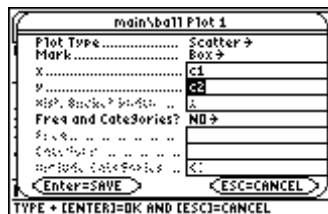
Make a scatter plot of the data and compare the solution found in Example 4 with the data.

**Solution**

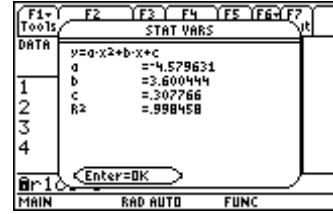
1. Press **[2nd] [F6] Clean Up** and select **2:NewProb** to clear variables and set other defaults.
2. Start the Data/Matrix Editor by pressing **[APPS]** and selecting **6:Data/Matrix Editor**. Then select **3:New**.
3. Move the cursor down to **Variable** and enter a name for the data variable (**ball**).
4. Press **[ENTER]** twice to display the Data/Matrix Editor. Enter the data for time in column 1 (*c1*) and the data for height in column 2 (*c2*). The headings above *c1* and *c2* are optional.



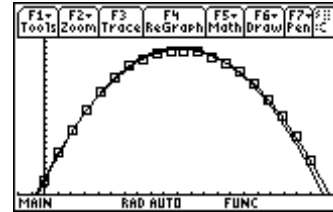
5. Set up the scatter plot. Press **F2** **Plot Setup** to display the main\ball dialog box. Press **F1** to define the plot. For the **Plot Type**, press **1** and select **1:Scatter**. For **Mark**, select **1:Box**. Enter **c1** and **c2** for  $x$  and  $y$ .
6. Press **ENTER** twice to return to the Data/Matrix Editor.
7. Press **Y=** to display the Y = Editor. In  $y1$ , enter the equation found with **deSolve** in Example 4. Use  $x$  in place of  $t$ .
8. Press **F2** **Zoom** and select **9:ZoomData** to see the match between the equation and the data.
9. Finally, find a quadratic regression equation for the data and compare it with the equation found with **deSolve**. Press **APPS** **6:Data/Matrix Editor** and then select **1:Current** to return to the Data/Matrix Editor.
10. Press **F5** **Calc** to display the main\ball Calculate dialog box. Press **9** and select **9:QuadReg**.
11. Enter **c1** and **c2** for  $x$  and  $y$ . Then move to the **Store RegEQ to** (store regression equation) menu, press **Y2(x)**.



12. Press  $\boxed{\text{ENTER}}$  to calculate and store the regression equation. Then press  $\boxed{\text{ENTER}} \boxed{\blacktriangleright} \boxed{\text{GRAPH}}$  to see and compare the graph of the regression equation with the graph of the equation found with **deSolve**.



The two equations match fairly well.



### Example 6: Newton's Law of Cooling

This example uses **deSolve** to solve a problem involving Newton's Law of Cooling. Newton's law describes the rate at which an object cools when it is immersed in surroundings that are colder than the object. It says the rate at which the object's temperature changes is directly proportional to the difference between the temperature of the object and the temperature of the surrounding medium.

If  $y$  is the temperature of an object over time  $t$  and  $t_s$  is the temperature of the surroundings, then Newton's law says

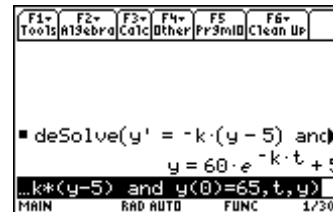
$$\frac{dy}{dt} = -k(y - t_s).$$

A temperature probe is connected to a Texas Instruments CBL™. The probe is heated to a temperature of 65°C. It is placed in water that has a temperature of 5°C. The probe cools to a temperature of 11°C in 30 seconds. Predict the temperature 60 seconds after the probe was placed in the cold water.

#### Solution

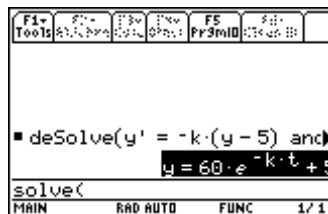
1. Press  $\boxed{2\text{nd}} \boxed{\text{F6}}$  **Clean Up** and select **2:NewProb** to clear variables and set other defaults.
2. Enter the **deSolve** command.

$\boxed{\text{CATALOG}} \boxed{\text{deSolve}} \boxed{(Y \ 2\text{nd}} \boxed{,} \boxed{)} \boxed{=}$   $\boxed{(-)} \boxed{\text{K}} \boxed{\times}$   $\boxed{(} \boxed{Y} \boxed{-} \boxed{5} \boxed{)}$   
 $\boxed{\text{CATALOG}} \boxed{\text{and Y}} \boxed{(} \boxed{0} \boxed{)} \boxed{=}$   $\boxed{65}$   $\boxed{,}$   $\boxed{\text{T}}$   $\boxed{,}$   $\boxed{\text{Y}}$   $\boxed{)}$



3. Now use the result and final conditions in the **solve**( command to find the constant of proportionality  $k$ .

**CATALOG** **solve**(  $\odot$  **ENTER**  $\leftarrow$  **K**  $\leftarrow$  **T**  $\equiv$  **30** **CATALOG** **and** **Y**  $\equiv$  **11** **ENTER**

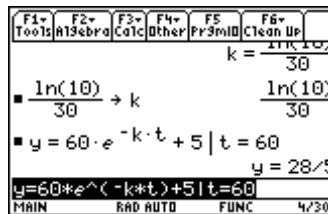
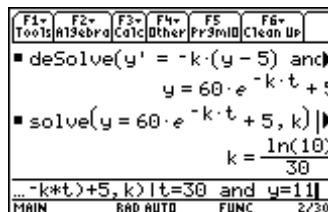


4. Store this value in  $k$ .

**2nd** **[LN]** **10**  $\div$  **30** **STO**  $\rightarrow$  **K**

5. Evaluate the equation from the result of step 2 for temperature of  $t=60$ .

$\odot$   $\odot$   $\odot$   $\odot$   $\odot$  **ENTER** **T**  $\equiv$  **60** **ENTER**



The temperature should be about  $28/5$  or  $5.6^\circ\text{C}$  after 60 seconds.

### Example 7: Resistance proportional to velocity

A ball is dropped from a height of 75 meters. Assume the acceleration due to gravity is  $-9.8 \text{ m/s}^2$  and deceleration due to air resistance is directly proportional to velocity with the constant of proportionality equal to 0.05. When will the ball hit the ground?

#### Solution

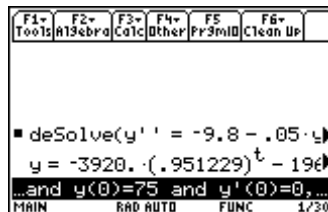
The distance traveled can be found by solving the differential equation

$$y'' = -9.8 - .05y' \text{ with initial conditions}$$

$$y(0) = 75 \text{ and } y'(0) = 0.$$

- Press **2nd** **[F6]** **Clean Up** and select **2:NewProb** to clear variables and set other defaults.
- Enter the **deSolve**( command using the equation and initial conditions.

**CATALOG** **deSolve**( **2nd** **[']** **2nd** **[']**  $\equiv$  **(-)** **9.8** **-** **.05Y** **2nd** **[']** **CATALOG** **and** **Y**  $\leftarrow$  **0**  $\equiv$  **75** **CATALOG** **and** **Y** **2nd** **[']**  $\leftarrow$  **0**  $\leftarrow$  **T**  $\leftarrow$  **Y**



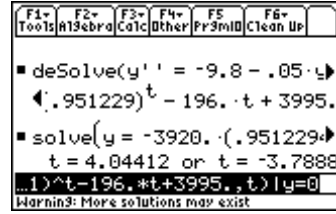


The expression for distance is

$$y = -3920(.951229)^t + 196t + 3995.$$

- Use this result with the **solve(** command to solve for  $t$  with  $y = 0$ .

The ball will hit after about 4.04412 seconds.



### Example 8: Logistic growth

In the last example, you will solve a logistic growth problem. In logistic growth problems, assume that the rate of growth of a population ( $k$ ) is directly proportional to both the population ( $y$ ) and the carrying capacity ( $C$ ) minus the population.

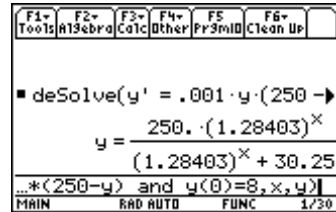
$$\frac{dy}{dt} = ky(C - y)$$

Eight wolves are introduced into a national park. Assume zoologists have determined a carrying capacity of 250 wolves and a growth rate constant of .001. When will the population reach 100?

#### Solution

- Press **2nd** **[F6]** **Clean Up** and select **2:NewProb** to clear variables and set other defaults.
- Use **deSolve(** to solve the differential equation. Let  $x$  equal time so that you can graph the solution.

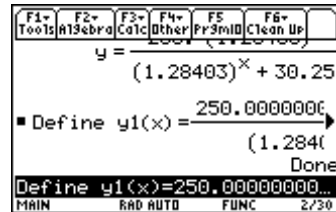
**CATALOG** **deSolve**(**Y** **2nd** **[']** **=** **.001Y** **x** **(** **250** **-** **Y** **)**  
**CATALOG** and **Y** **(** **0** **)** **=** **8** **,** **X** **,** **Y** **)**



- Define  $y1(x)$  to be the solution from step 2.

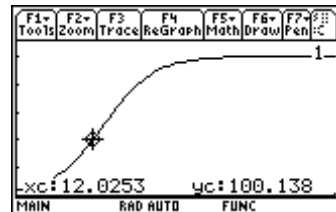
**CATALOG** **Define** **Y1** **(** **X** **)** **=** **↵** **ENTER**

Delete **y=** from the expression, and press **ENTER**.



- Set the Window variable values for a  $[0,50] \times [0,275]$  viewing window.
- Graph  $y1(x)$  and trace (**F3**) until the  $y$ -coordinate is about 100.

The population will reach 100 in about 12 years.



**Exercises**

1. Find the area between the curves

$$y = \sin(x^2) \text{ and } y = 1 - x^2.$$

2. Find the length of the curve
- $y = \sin^{-1} x$
- for
- $-1 \leq x \leq 1$
- . Use the Arc feature and compare the answer with the result of the definite integral that gives arc length.

Use **deSolve**( to solve the following differential equations (exercises 3-6).

3.  $y' = x^2 \cos(x)$ ,  $y(0) = 0$

4.  $v' = -\frac{k}{m}v$ ,  $v(0) = v_0$

5.  $y'' = -32$ ,  $y(0) = 19$ ,  $y'(0) = 0$

6.  $q'' = 1 - 2q - 2q'$ ,  $q(0) = 0$ ,  $q'(0) = 0$

7. A book was dropped from a height of .8649 meters. Its height above ground was measured with a Texas Instruments CBL™. Here are the data:

Time (seconds)	Height (meters)
0	.8649
.02	.8583
.04	.8484
.06	.8364
.08	.8188
.10	.7991
.12	.7749
.14	.7475

Time (seconds)	Height (meters)
.16	.7156
.18	.6805
.20	.6421
.22	.6004
.24	.5543
.26	.5060
.28	.4544

- (a) Solve the second-order differential equation

$$y'' = -9.8$$

to find an equation to model the height of this book over time.

- (b) Make a scatter plot of the data and graph the solution from part (a) with this scatter plot.
8. A cup of hot chocolate is left to cool on a kitchen table. It cools from 93°C to 55°C in 15 minutes. If the room temperature is 21°C, predict the temperature of the hot chocolate 20 minutes after it is placed on the table.
9. A projectile is fired straight up with an initial velocity of 87m/s. Assume the acceleration due to gravity is  $-9.8 \text{ m/s}^2$  and deceleration due to air resistance is directly proportional to velocity with the constant of proportionality equal to 0.05. When will the projectile hit the ground?
10. A rumor spreads through a school with 1500 students. If 4 students initially hear the rumor and the growth rate is .00025, use a logistic model to predict how long it will take for the rumor to spread to 1000 students.