Newton's Law of Cooling and The Calculus Behind It

Brenda Batten, T^3 Instructor

The rate at which an object's temperature is changing at any given time is proportional to **the difference** between its temperature and the temperature of the surrounding medium.

T = Temperature of the cooling (or warming) object
t = time in seconds since the first reading
T_S = Temperature of surrounding medium

A. Using Calculus to Derive the Model

This is an example of exponential growth and decay as follows:

$$\frac{dy}{dt} = Ky$$

$$\frac{1}{y}dy = Kdt$$

$$\int \frac{1}{y}dy = \int kdt$$

$$\ln|y| = Kt + C$$

$$|y| = e^{Kt + C}$$

$$|y| = e^{Kt} \cdot e^{C}$$

at $t = 0$, $y = y_0$ $\qquad |y_0| = e^{0} \cdot e^{C} = e^{C} \quad \Rightarrow \quad y_0 = \pm\, e^{C}$

$$y = y_0 \cdot e^{Kt}$$

Let a chilled object warm (or a hot object cool) to room temperature. Then by Newton's Law, the rate at which the object's temperature is changing at any given time is proportional to **the difference** between its temperature and the temperature of the surrounding medium. Consider the following differential equation:

$$\frac{dy}{dt} = \frac{d(T - T_S)}{dt} = \frac{dT}{dt} - \frac{dT_S}{dt} = \frac{dT}{dt} - 0 = \frac{dT}{dt} = k(T - T_S)$$

Let $y = T_S - T$ or $T - T_S$, whichever is a positive quantity.

This quantity approaches zero since the temperature of the object approaches the ambient temperature, i.e. the temperature of the surrounding air.

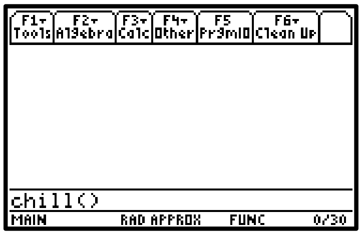
Using $y = T_S - T$ and $y_0 = T_S - T_o$, the equation $y = y_0 \cdot e^{Kt}$ becomes $T_S - T = (T_S - T_0)\, e^{KT}$

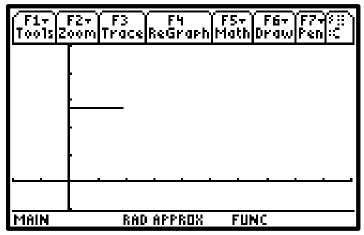
The exponential model, then, models the quantity y, which is defined to be the difference between the room temperature and the temperature of the cooling (or warming) object at any given time.

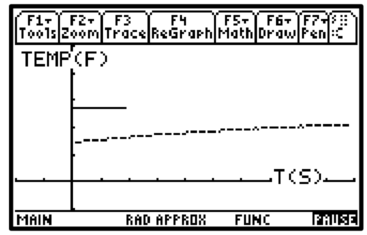
B. Collecting the Data

Temperature data can be stored in the TI-89 using the CBL program CHILL89. It stores time (in seconds) in List 1, l1, and the temperature readings (in Fahrenheit) in List 2, l2.

[2nd] [-] VAR-LINK: Highlight CHILL89
[ENTER]: Pastes program name onto the command line of the home screen.
[)]: Close parenthesis
[ENTER]



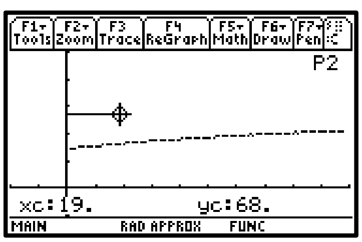


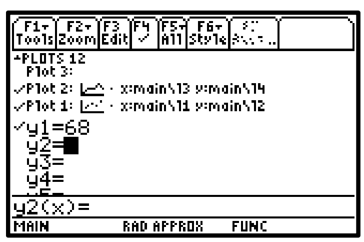


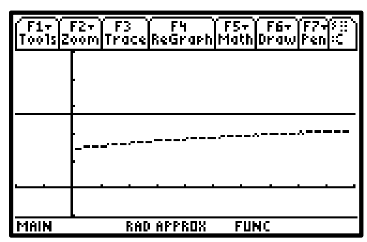
First use the CBL to determine the room temperature, T_S. Then collect temperature data following the directions within the program. Place the thermometer in ice (or hot) water for a few seconds. Then remove the thermometer and take data readings as it warms (or cools) to room temperature. Finally, from the menu, select QUIT, then [ENTER] to return to the home screen.
◊ [F3] Graph

Notice that two Plots have been defined. In Plot 1, x= List 1, time in seconds, and y= List 2, temperature of the warming (or cooling) object. In Plot 2, x= List 3, time in seconds and y= List 4, room temperature.

[F3]: Trace on plot 2 to determine room temperature, T_s.
◊ [F1] Y=: Let y1 = T_S.
◊ [F3] Graph







C. Model the Data

In order to model the data, use the exponential equation $y = a * b^x$. Again, the positive quantity y represents the **differences** of room temperature and temperature of the warming (or cooling) object at any given time.

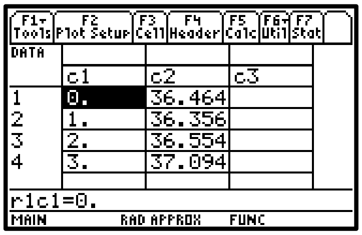
For example, using temperatures of ice water warming to room temperature, let $y = T_S - T$, so that y is a positive quantity. (For temperature of hot water which cools to room temperature, let $y = T - T_S$ and proceed accordingly.)

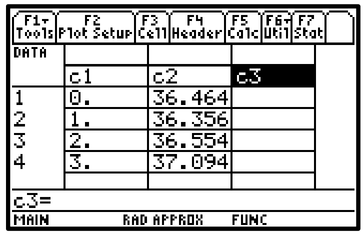
1. Calculate Differences

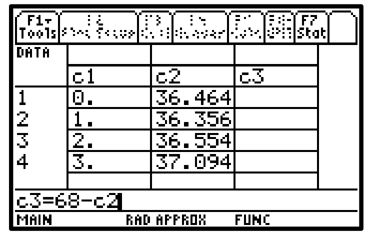
View the Data:

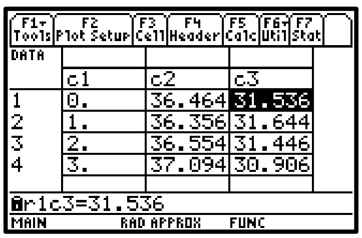
[APPS] 6: Data/Matrix Editor
1: Current
c1 contains List 1, l1, time in seconds
c2 contains List 2, l2, temperature data.

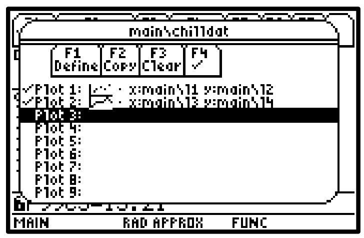
In the Data/Matrix Editor, let $y = T_S - T$, for ice water warming to room temperature
define $c3 = T_S - c2$, since T is stored in c2 .

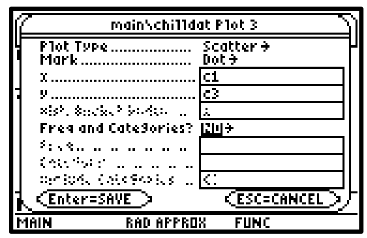






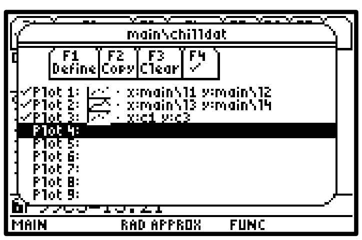


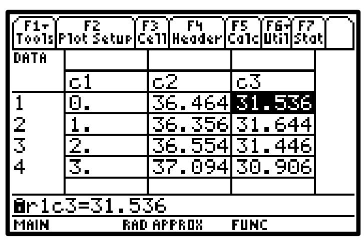


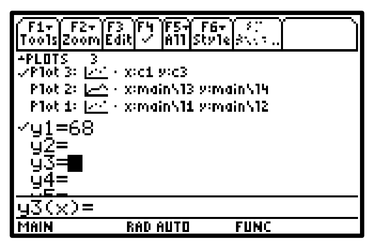


[F2] Plot Setup: Highlight Plot 3. [F1] Define:

Plot Type: Scatter
Mark: Dot
x: c1 y: c3
Freq and Categories? No

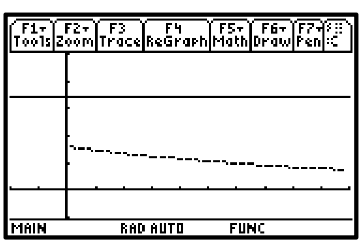






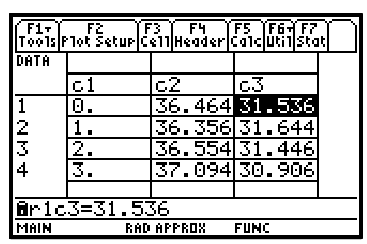
[Enter=Save] [Enter] Return to Data/Matrix Editor

◊ [F1] Y=
[F4] Deselect Plots 1 and 2



◊ [F3] Graph

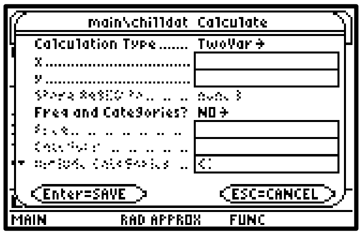
Return to Data Matrix Editor

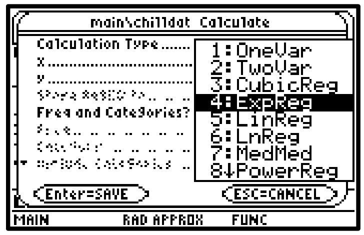


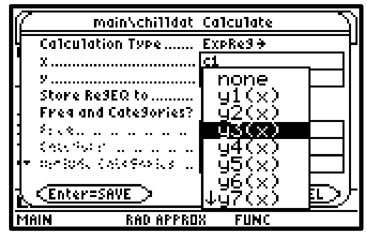
[APPS] 6: Data/Matrix Editor
1: Current

2. Calculate Exponential Regression Equation for Differences

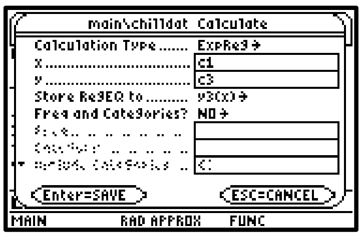
From the Data/Matrix Editor, use the Calculate menu to calculate the regression equation.

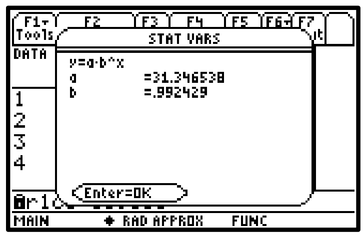


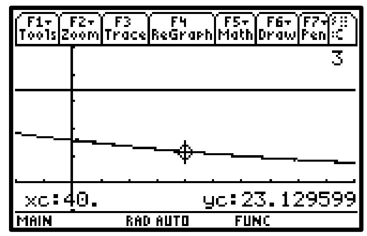




[F5] Calculate Type: 4: ExpReg: x = c1, y = c3 Store RegEQ to y3(x)







Freq & Categories: NO [Enter=Save] [Enter=OK] ◊ [F3] Graph [F3] Trace

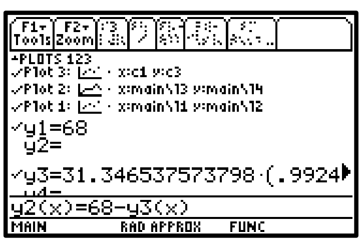
3. Derive the Equation that Models the Original Data

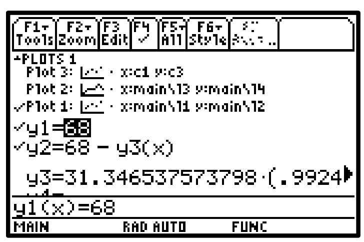
The quantity we have modeled is for the differences: y3 = c3 = T_S - c2 .
To retrieve the original data, solve for c2.

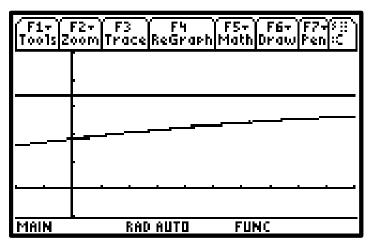
$$y3 = a \cdot b^x$$

$$T_S - c2 = (T_S - T_0)\, e^{Kt} \quad \text{which is stored in } y3.$$

$$c2 = T_S - (T_S - T_0)\, e^{Kt}$$





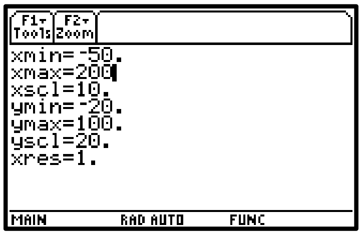


◊ [F1] Y= [F4] Select only Plot 1, y1, y2 ◊ [F3] Graph
Define y2 = T_S - y3(x). y3 models the data collected.

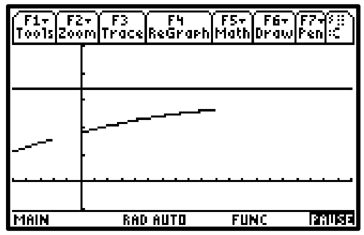
4. Applications of Modeling

One application of modeling data is to extrapolate information that is beyond the domain of the collected data. For example, this CBL program collects data over a time period of 98 seconds. Our model, however, can give us a prediction of what the temperature will be at any point in time.

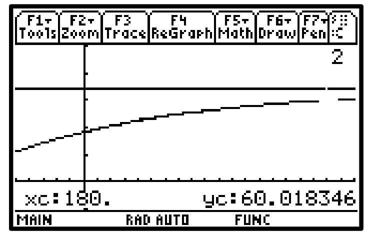
To illustrate this, expand the viewing window of the plotted graph.



◊ [F2] Window:
xmin = -50 and xmax = 200



◊ [F3] Graph



[F3] Trace on graph of y2
180 [ENTER]

Move the cursor up or down until you are tracing on the graph of y2. Then enter 180. This moves the cursor to the position t=180 seconds.

What will the temperature be 3 minutes after the probe was removed from the ice water? The y-coordinate gives the predicted temperature of the probe at that time according to the model we have derived.

The CBL program “Chill89” is shown below for your reference. It is an adaptation of “Chill” from the workbook “Real World Math with the CBL System”. The portion of this program for seeing directions is not included. Therefore, do not choose the menu selection **3: Collect (Dir).** Menu selection **4: Use Sample** gives the data that was used in this handout. Choosing this option stores data in lists 1, 2, 3, and 4 as discussed in the handout above.

```
()
Prgm
DelVar  sysData
©clear various screens that will be used
ClrHome:ClrIO
ClrGraph:ClrDraw

Local  premode
getMode("ALL")→premode

©Standard calc setup
PlotsOff
FnOff

setMode({"Graph","Function","Exponential
Format","Normal","Exact/Approx","Approximate","Split
Screen","Full","Display Digits","Float"})

setGraph("Coordinates","Rect")
setGraph("GraphOrder","Seq")
setGraph("Grid","Off")
setGraph("Labels","Off")
setGraph("Axes","Off")

Ø→xmin
159→xmax
⁻99→ymin
Ø→ymax
```

```
©The cover page
PxlLine  3,3,3,155
PxlLine  3,155,19,155
PxlLine  19,155,19,3
PxlLine  19,3,3,3
PxlText  "Real-World Math with CBL",8,6
PxlText  "Texas Instruments",27,3Ø
PxlText  "CHILL OUT (v2.Ø)",38,4Ø
PxlText  "(Activity #1Ø)",63,41
PxlText  "Press [ENTER] to continue",53,7
Pause
ClrHome:ClrGraph

Lbl  menu
ClrIO:ClrDraw:ClrHome
Local  opt,i,status,linkmsg
1→opt
Disp  "","   *** OPTIONS ***     ",""
PopUp      {"Room   Temp","Collect   (No   Dir)","Collect   (Dir)","Use
sample","QUIT"},opt

If opt=5 Then: Goto  end:EndIf

If opt=4 Then
©DATA1Ø
Goto  data1Ø
EndIf

If opt=3 Then
Disp  ""," ***This option not available***"
Goto  menu
EndIf

If opt=2 Then
©clear various screens that will be used
ClrHome:ClrIO
ClrDraw
   Goto  part2
EndIf



If opt=1 Then
©clear various screens that will be used
ClrHome:ClrIO
ClrDraw
⁻25→ymin
125→ymax
25→yscl
⁻5→xmin
25→xmax
5→xscl

Send  {1,Ø}
Send  {1,1,11}
newList(2Ø)→l4
```

```
Disp  "Collect room temp"
Disp  "Hit [ENTER]"
Pause
ClrIO
ClrHome:ClrGraph
ClrDraw
PxlText  "TEMP(F)",3,2
PxlText  "T(S)",58,125
PlotsOff
Send  {3,Ø.5,⁻1,Ø}
For  i,1,2Ø,1
Get  l4[i]
PtOn  i,l4[i]
EndFor
seq(n,n,Ø,19,1)→l3
Lbl  at
FnOff
setMode("Graph","Function")
setGraph("Axes","On")
ClrIO
⁻25→ymin
125→ymax
25→yscl
⁻2Ø→xmin
1ØØ→xmax
1Ø→xscl
NewData  roomdat,l3,l4
NewPlot  2,2,l3,l4,,,,5
DispG
PxlText  "TEMP(F)",3,2
PxlText  "T(S)",53,12Ø
Pause
StoGDB  gdb5
Ø→u
Ø→v
Goto  menu
EndIf


If opt=2 Then
©clear various screens that will be used
ClrHome:ClrIO
ClrGraph:ClrDraw

1→linkmsg
ClrIO
Goto  linkchk
Goto  menu
EndIf

Lbl  part2
©clear various screens that will be used
ClrHome:ClrIO
ClrGraph:ClrDraw

⁻25→ymin
```

```
125→ymax
25→yscl
⁻2Ø→xmin
1ØØ→xmax
1Ø→xscl
Send  {1,Ø}
Send  {1,1,11}
newList(99)→l2
Disp  "Place the probe"
Disp  "in hot or cold water"
Disp  "for about a minute."
Disp  "","Hit [ENTER]"
Pause
ClrIO
Disp  "Remove the probe"
Disp  " from the cup."
Disp  ""
Disp  "Hit [ENTER] to graph"
Disp  "temperature."
Pause
ClrDraw
PxlText  "TEMP(F)",3,2
PxlText  "T(S)",58,125
Send  {3,Ø.5,⁻1,Ø}
For  i,1,99,1
Get  l2[i]
PtOn  i,l2[i]
EndFor
seq(n,n,Ø,98,1)→l1
Lbl  as
FnOff
setMode("Graph","Function")
setGraph("Axes","On")
ClrIO
⁻25→ymin
125→ymax
25→yscl
⁻2Ø→xmin
1ØØ→xmax
1Ø→xscl
NewData  chilldat,l1,l2
NewPlot  1,1,l1,l2,,,,5
DispG
PxlText  "TEMP(F)",3,2
PxlText  "T(S)",58,12Ø
Pause
StoGDB  gdb6
Ø→u
Ø→v
Goto  menu
Goto  end


Lbl  data1Ø
{Ø,1,2,3,4,5,6,7,8,9,1Ø,11,12,13,14,15,16,17,18,19,2Ø,21,22,23,24,25,26
,27,28,29,3Ø,31,32,33,34,35,36,37,38,39,4Ø,41,42,43,44,45,46,47,48,49,5
```

```
Ø,51,52,53,54,55,56,57,58,59,6Ø,61,62,63,64,65,66,67,68,69,7Ø,71,72,73,
74,75,76,77,78,79,8Ø,81,82,83,84,85,86,87,88,89,9Ø,91,92,93,94,95,96,97
,98}→l1
{36.464,36.355999999999,36.553999999999,37.Ø94,37.364,37.31,37.562,38.1
Ø2,38.Ø47999999,9999,38.3,38.57,38.822,39.Ø92,39.343999999999,39.56,39.
812,4Ø.Ø82,4Ø.334,4Ø.586,4Ø.838,41.Ø9,41.342,41.612,41.864,42.116,42.11
6,42.Ø62,42.314,42.854,43.1Ø6,43.358,43.322,43.556,43.8Ø8,43.8Ø8,44.Ø6,
44.312,44.546,44.546,44.798,45.Ø5,45.283999999999,45.536,45.536,45.77,4
6.ØØ4,46.ØØ4,46.255999999998,46.49,46.742,46.742,46.976,46.976,47.21,47
.462,47.696,47.696,47.93,48.164,48.164,48.398,48.398,48.38,48.884,48.88
4,49.118,48.847999999999,49.352,49.586,49.586,49.82,49.82,5Ø.Ø54,5Ø.Ø54
,5Ø.27,5Ø.5Ø4,5Ø.5Ø4,5Ø.486,5Ø.486,5Ø.972,5Ø.972,51.2Ø6,5Ø.953999999999
,51.188,51.188,51.4Ø4,51.4Ø4,51.637999999999,51.872,51.872,51.872,52.1Ø
6,52.1Ø6,52.322,52.322,52.556,52.79,52.79}→l2
{Ø,1,2,3,4,5,6,7,8,9,1Ø,11,12,13,14,15,16,17,18,19}→l3
{68.36,68.162,68.179999999999,68.179999999999,67.982,67.982,68.17999999
9999,68.179999999999,67.982,67.982,67.982,67.982,67.982,68.197999999999
,68.197999999999,68.197999999999,68.197999999999,68,68,68}→l4
FnOff
setMode("Graph","Function")
setGraph("Axes","On")
ClrIO
⁻25→ymin
125→ymax
25→yscl
⁻2Ø→xmin
1ØØ→xmax
1Ø→xscl

NewData  sample,l1,l2,l3,l4
NewPlot  1,1,l1,l2,,,,5
NewPlot  2,1,l3,l4,,,,5
DispG
PxlText  "TEMP(F)",3,2
PxlText  "T(S)",58,12Ø
Pause
StoGDB  gdb6
Ø→u
Ø→v
Goto  menu
Goto  end

Lbl  end
setMode(premode)
setMode("Split 1 App","Window Editor")
setMode("Split 1 App","Home")

EndPrgm
```